

Chapter 7

Singular optics and superresolution

Objects with a vortex structure exist in the various spheres of the material world, in the macrocosm (the spiral shape of galaxies and nebulae), in the microcosm (elementary particles, optical fields) and in our daily lives (cyclones and anticyclones, tornadoes and typhoons). Their structure and behaviour have still not been exhaustively studied and represent a vast field for research. Thus, in recent years a separate section ('singular optics') was formed by the branch of optics dealing with the study of light beams with screw phase singularities (i.e. the vortex laser beams).

At the point of singularity the intensity of the light field is zero, and the phase is undefined. There are abrupt phase changes in the vicinity of this point.

Singular features in light fields may appear as they pass through randomly inhomogeneous and nonlinear media. It is also possible to excite the vortex fields in laser resonators and multimode optical fibres. The simplest and most controllable method of forming the vortex fields is to use spiral diffractive optical elements (DOE), and dynamic liquid-crystal transparents (energy efficiency of the latter is still quite low). The simplest of such DOEs are spiral phase plates (SPP) and helical axicon (HA).

Vortex laser beams are the subject of numerous studies and publications by Russian scientists and their foreign counterparts. The properties of such beams on the basis of the Bessel, Laguerre–Gaussian (LG), Gauss–Hermite (GE) modes, etc. are being actively studied [1–3].

The scope of optical vortices is constantly expanding. In particular, in nanophotonics problems it is proposed for use them to manipulate dielectric micro- and nano-objects. For example, a recent paper [4] studied the motion of gold nanoparticles (100 nm to 250 nm), captured in the central part of the optical vortex. Also, more attention is paid to exploring the possibilities of using plasmon effects in nano-tweezers [5, 6].

The use of optical vortices in photolithography can achieve a resolution of $\lambda/10$ (λ is the wavelength of light). It is possible to use efficiently optical spiral structures, even with a small number of quantization levels [7].

Other applications include optical vortices, such as interferometry: using the SPP, placed in the plane of the spatial spectrum of a $4f$ -optical system (f is the focal length of the spherical lens) a method was proposed for producing spiral interferograms which can be used to easily distinguish convex and concave sections of the wavefront [8].

Spiral filters are used for contrasting and relief imaging of nanosized phase objects [9].

SPPs are also used in stellar coronagraphs, in which the light from a bright star is converted into the ring and stopped down, and the faint light from the planets of the star passes through the aperture and registers. It is known that vortex waves in a coherent system have a well-defined phase, which, however, are poorly defined in the partially-coherent system. In the limit, for a fully incoherent case neither the helical phase nor zero intensity is observed. This allows the optical vortices to be excluded from the observation of coherent radiation in order to enhance the incoherent signal; this effect is used in coronagraphs [10, 11].

SPPs are also used for the optical realisation of the radial Hilbert transform [12]. Hilbert-optics, as well as shadows optics with transformations taking place in a similar frequency domain, has been used successfully for pre-processing of images and phase analysis. Hilbert spectroscopy allows to achieve nanoresolution for spectral analysis. Using the radial Hilbert transform, including fractional-based transform, SPP opens up new possibilities in solving the problems mentioned above.

Phase dislocations that define zero intensity, represent a promising tool in metrology. Since the accuracy of determining the position of the dislocation is not limited to the classical diffraction limit (the gradient of the phase change in this case tends to infinity), and is limited only by the signal / noise ratio, the geometry of an object subject to the availability of a priori information about the object can be determined with very high accuracy [13]. This approach is based on the method of optics-vortex metrology [14], successfully applied in the optical vortex interferometer, which allows to track objects with nanometer displacement accuracy [15].

The sensitivity of singular beams to wavefront changes and all kinds of defects can be used to test surfaces [16] and for the analysis of optical systems [17].

With the help of optical vortex interferometers, which are based on the generation of light fields, representing regular gratings or grids of optical vortices [18, 19] (i.e. measurements are carried out of the status of nodes with not the maximum but minimum light intensity) we can determine the angles rotation with an accuracy of 0.03 arc seconds [20] and measure the angles of inclination of the wavefront with an accuracy of 0.2 arc seconds [21].

In non-linear optical media, optical vortices can be used to form waveguide structures [22] and ‘labyrinths’ [23], and also to study various physical phenomena [24, 25].

This chapter describes the main types of paraxial optical vortices and their formation by diffractive optical elements in the scalar theory of diffraction. Vector diffraction is studied for the SPPs.

7.1. Optical elements that form wavefronts with helical phase singularities

Consider the light fields having a wavefront with a helical phase singularity. The complex amplitude of such fields is as follows:

$$E(r, \varphi, z) = A(r, z) \exp(in\varphi), \quad (7.1)$$

where (r, φ, z) are the cylindrical coordinates, n is the order of the phase singularity or topological charge.

These fields are called optical vortices, they have an orbital angular momentum, and the Umov–Poynting vector is directed along the helix (Fig. 7.1).

Optical vortices are formed by the spiral optical elements, which include a spiral phase plate (SPP) (Fig. 7.2a). Figure 7.2b shows a conical axicon. When combining the axicon and the SPP we obtain a helical or spiral axicon.

7.1.1. The spiral phase plate (SPP)

The spiral phase plate (SPP) as an optical element, whose transmittance function is proportional to $\exp(in\varphi)$, φ is the polar angle, n is an integer (the order of the SPP),

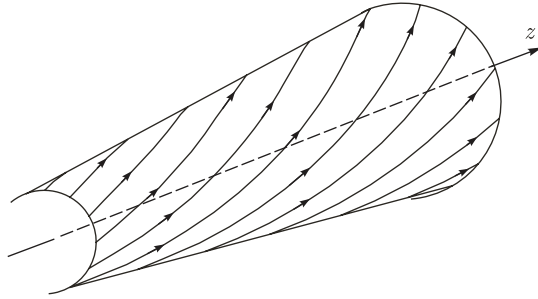


Fig. 7.1. The direction of energy transfer in optical vortices (z is the axis along which light propagates).

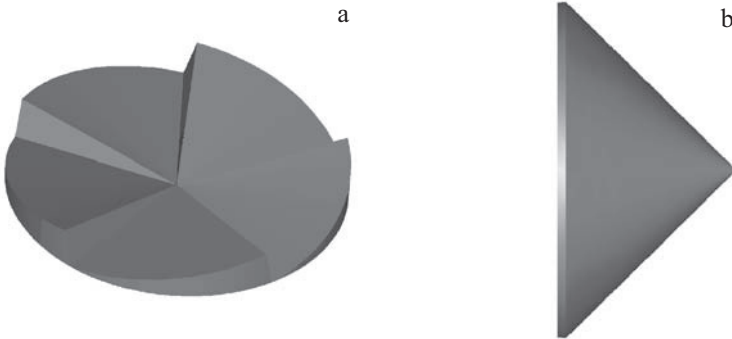


Fig. 7.2. The spiral phase plate (a) and conical axicon (b).

was first produced and analyzed in [26]. In recent years, particularly with respect to the optical manipulation of microparticles, the interest in the SPP increased [27-32]. In [30] SPP few millimeters in diameter with $n = 3$ were produced and characterized at a wavelength of 831 nm using the moulding technology with a maximum height of the microrelief of 5 μm . The accuracy of manufacturing the surface relief of the SPP on a polymer was very high ($\sim 3\%$ error). In [27] using a conventional scanning electron microscope, converted into an electron-beam lithograph, writing directly on the negative photoresist SU-8 was carried out to record an SPP with a diameter 500 μm with $n = 1$ and with a continuous profile of relief with a maximum height of the step of 1.4 μm for a helium–neon laser. The formed diffraction pattern of a Gaussian beam on the SPP differed from the ideal ‘doughnut’ form by only 10%. The SPP made in this way was used for simultaneous optical trapping of latex beads of diameter 3 μm each, with the refractive index $n' = 1.59$. In [29] the same authors showed that the displacement of the centre of the SPP from the axis of the Gaussian beam results in the formation of an off-axis vortex and its transverse intensity distribution is rotated around the optical axis during beam propagation.

In [31] using standard photolithographic techniques with four binary amplitude masks (reticles) the authors constructed and studied a 16-tier and 16-sector SPP for a pulsed solid-state laser with a wavelength of 789 nm. SPP was made on SiO_2 100 mm in diameter with a maximum height of the step of the relief of 928 nm. In [28] using direct writing by the electron beam on a negative photoresist at a wavelength of 514 nm the authors produced an SPP with a diameter of 2.5 mm and a relief depth of 1082 nm. In addition, in [28] a theoretical analysis of Fresnel diffraction for a plane wave and Gaussian beam on the SPP was carried out. In [32], using a liquid-crystal spatial light modulator and a neodymium laser with doubled frequency and a wavelength of 532 nm an SPP with a high order singularity $n = 80$ was produced. In addition, in [32] analytical expressions were derived for Fraunhofer diffraction of the Gaussian beam on the SPP with large orders of singularity $n \gg 1$.

7.1.2. Spiral zone plates

The wavefronts with a helical phase singularity can also be produced by methods of digital holography [33]. When encoding a spiral phase plate with a circular carrier spatial frequency we obtain a function of the form $\text{sgn} [\cos (n\varphi + kr^2)]$. The transmission function of such a hologram is shown in Fig. 7.3.

7.1.3. Gratings with ???

When using the carrier frequency the hologram will have the form shown in Fig. 7.4.

7.1.4 Screw the conical axicon

The optical element called the axicon has been known for a long time [34]. It is a glass cone, which is illuminated from the base, and its optical axis passes along the



Fig. 7.3. The formation of optical vortices by methods of digital holography.

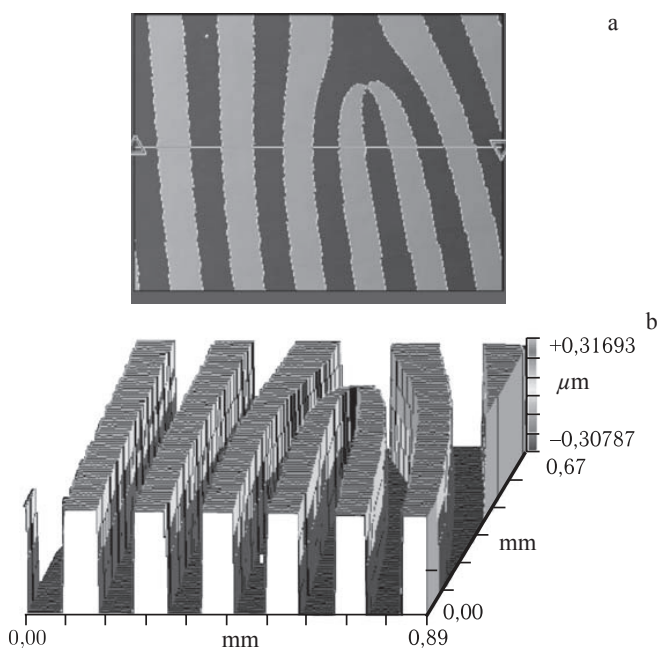


Fig. 7.4. A hologram with a one-dimensional spatial carrier frequency for generation of optical vortices.

height of the cone. It is usually used in optics to create a narrow ‘diffractionless’ laser beam [35, 36] or in conjunction with a lens to form a narrow annular light intensity distribution [37–39]. The diffractive axicon is shown in Fig. 7.5.

Axicons are also used in imaging systems to increase the depth of field, which can be 10–100 times greater than in a traditional lens.

In [40] grayscale photolithography was used to make on a low-contrast photoresist an optical element whose transmittance is proportional to the product

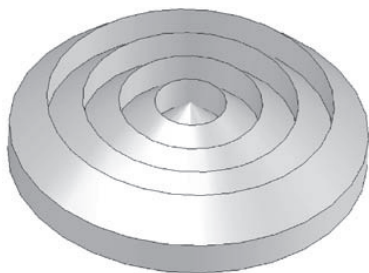


Fig. 7.5. Diffractive axicon.

of the transmission function of the axicon and SPP. This element is in [40] named ???, i.e. forming a light pipe. Such an optical element is sometimes called the helical axicon [41]. Diffraction of a plane wave on such an diffraction element is identical to the diffraction of a conical wave on the SPP. In [42] using a 16-level helical axicon of the 5-th order and 6 mm diameter, made by direct writing with the electron beam and a He–Ne laser experiments were carried out with optical trapping and rotation with a period of 2 s of yeast particles and polystyrene beads with a diameter of 5 μm .

7.1.5. Helical logarithmic axicon

In the geometric approximation, a special feature of the classical (conical) axicon is a linear increase of intensity on the optical axis [34, 43, 44].

In [45], attention was paid to a generalized axicon that generates a given intensity distribution along the optical axis. In particular, a logarithmic axicon is suitable for producing uniform intensity along the optical axis (Fig. 7.6).

7.2. The spiral phase plate

In this section, we discuss some of the early uses of spiral phase plates for optical information processing, namely the optical performance of the Hankel transform and radial Hilbert transform. The remainder of the section is devoted to the theory

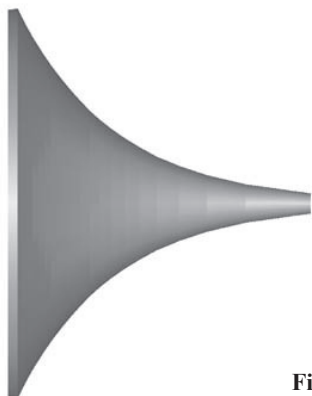


Fig. 7.6. Logarithmic axicon.

of diffraction of light by the SPP, it gives expressions for the diffraction of a Gaussian beam and plane wave on SPP in both the scalar approximation and taking into account the vector nature of the electromagnetic field.

7.2.1. Hankel transformation

In problems with cylindrical symmetry, for example, to create such images, like a circle, ring, or a set of rings [46] or by focusing on the 3D surface of revolution [47], to generate the laser modes [48], and in the formation of vortex beams [42], the calculations can be greatly accelerated by reducing the integral expressions to the Hankel transform.

For example, the Fourier transform in polar coordinates:

$$F(\rho, \theta) = -\frac{ik}{2\pi} \int_0^\infty \int_0^{2\pi} f(r, \varphi) \exp\left[-i\frac{k}{f_L} r \rho \cos(\theta - \varphi)\right] r dr d\varphi, \quad (7.2)$$

where (r, φ) and (ρ, θ) are the polar coordinates in the front and back focal plane of the lens, f_L is the focal length of the spherical lens used for the Fourier transform for the input function that is presented in the form $f(r, \varphi) = t(r) \exp(im\varphi)$, reduces to the Hankel transform of the m -th order:

$$H_m(\rho, \theta) = \exp(im\theta) \int_0^\infty t(r) J_m\left(\frac{kr\rho}{f}\right) r dr, \quad (7.3)$$

where $J_m(x)$ is the Bessel function of the first kind and m -th order:

$$J_m(z) = \frac{(-i)^m}{2\pi} \int_0^{2\pi} \exp(iz \cos \varphi + im\varphi) d\varphi. \quad (7.4)$$

A similar expression is obtained for the transformation of the Fresnel function describing the propagation of function $f(r, \varphi)$ in free space at distance z :

$$F_m(\rho, \theta, z) = \frac{ik}{z} \exp(ikz) \exp(im\theta) \exp\left(\frac{ik\rho^2}{2z}\right) \int_0^\infty t(r) \exp\left(\frac{ikr^2}{2z}\right) J_m\left(\frac{kr\rho}{z}\right) r dr. \quad (7.5)$$

For a quick calculation of the Hankel transform (7.3) we can consider using an exponential change of variables [49]. This method assumes that with the exponential change of variables the Hankel transforms reduces to convolution, which can be calculated using the fast Fourier transform algorithm.

Indeed, after the change of variables

$$r = r_0 e^x, \quad \rho = \rho_0 e^y, \quad (7.6)$$

where r_0 and ρ_0 are constant, instead of (7.3) we obtain

$$\bar{H}(y) = r_0^2 \int_{-\infty}^\infty \bar{t}(x) S(x+y) e^{2x} dx, \quad (7.7)$$

where

$$\begin{aligned}\bar{t}(x) &= t(r_0 e^x), S(x+y) = J_m \left(\frac{k}{f} r_0 \rho_0 e^{x+y} \right), \\ \bar{H}(y) &= H(\rho_0 e^y) = H_m(\rho, \theta) \exp(-im\theta).\end{aligned}\quad (7.8)$$

For the function $S(x)$ to tend to 0 at $x \rightarrow \pm \infty$, it can be multiplied by $\exp(x/4)$, and to ensure that the integrand in (7.7) is not changed, function $\bar{t}(x)$ should also be multiplied by $\exp(x/4)$.

Since the transmission function $t(r)$ is limited by the aperture $r \in [0, R]$, where R is the radius of the aperture, no problems arise for function $\bar{t}(x)$ at $x \rightarrow \infty$.

We introduce the following notation:

$$t_c(x) = \bar{t}(x) r_0^2 e^{7x/4}, S_c(x+y) = S(x+y) e^{x+y/4}, H_c(y) = \bar{H}(y) e^{y/4}, \quad (7.9)$$

and instead of (7.7) we obtain the convolution

$$H_c(y) = \int_{-\infty}^a h_c(x) S_c(x+y) dx, \quad a = \ln \frac{R}{r_0}. \quad (7.10)$$

The integral (7.10) can be expressed in terms of 1D Fourier transform:

$$H(\rho) = \left[\frac{\rho}{\rho_0} \right]^{-\frac{1}{4}} \int_{-\infty}^{\infty} T(-u) U(u) e^{iu \ln \frac{\rho}{\rho_0}} du, \quad (7.11)$$

where $T(u)$ is the Fourier transform of the functions $h_c(x)$, $U(u)$ is the Fourier transform of $S_c(y)$.

There are other methods for rapid calculation of the Hankel transform [50].

When using the SPP as a filter in the spatial frequency plane of the Fourier correlator, we obtain the following chain of expressions. Before the filter frequency distribution of the radially symmetric function is a Hankel transform of zero order:

$$F(\rho) = 2\pi \int_0^{\infty} f(r) J_0(2\pi r \rho) r dr. \quad (7.12)$$

At the output of the correlator taking into account (7.4) and $J_m(-x) = (-1)^m J_m(x)$ we obtain:

$$\begin{aligned}E_c(r', \varphi') &= 2\pi \int_0^{\infty} \int_0^{2\pi} F(\rho) \exp(im\theta) \exp[-2\pi r' \rho \cos(\theta - \varphi')] \rho d\rho d\theta = \\ &= \frac{4\pi^2}{i^m} \int_0^{\infty} F(\rho) J_m(2\pi r' \rho) \rho d\rho = E_c(r').\end{aligned}\quad (7.13)$$

This is a radially symmetric function, equal to the Hankel transform of m -th order of the Fourier spectrum $F(\rho)$ of the original function $f(r)$.

7.2.2. Radial Hilbert transform

The Hilbert transform is used in image processing since it emphasizes the contour of objects. The disadvantage of this transformation is the one-dimensionality, and the detection of contours takes place along a single direction. In [51], attention was paid to the radially symmetric version of the Hilbert transform, allowing a two-dimensional detection of the contours of objects of arbitrary shape.

Let $g(x, y)$ be a function describing the light field in the input plane of the Fourier correlator. The convolution operation is performed in this correlator, and the light field with the complex amplitude of the form in the output plane is obtained

$$\tilde{g}(x, y) = g(x, y) * h(x, y), \quad (7.14)$$

where $h(x, y)$ is the Fourier transform of the function of the masks in the frequency domain of the correlator $H(u, v)$.

The mask for a one-dimensional Hilbert transform of the P -th order is as follows:

$$H_P(u) = \exp\left(\frac{iP\pi}{2}\right)S(u) + \exp\left(-\frac{iP\pi}{2}\right)S(-u), \quad (7.15)$$

where $S(u)$ is the Heaviside function (step function).

The function (7.15) can be rewritten as follows:

$$H_P(u) = \cos\left(\frac{P\pi}{2}\right) + i \sin\left(\frac{P\pi}{2}\right) \text{sgn}(u), \quad (7.16)$$

where $\text{sgn}(u)$ is the function of the mark.

Since the Fourier transform of the sign function $\text{sgn}(u)$ has the form $1 / (i\pi x)$, then by (7.14) in the output plane of the correlator is a field with amplitude

$$\tilde{g}(x, y) = g(x, y) \cos\left(\frac{P\pi}{2}\right) + i \sin\left(\frac{P\pi}{2}\right) \left[g(x, y) * \frac{1}{i\pi x} \right], \quad (7.17)$$

Such a transformation is still one-dimensional. To generalize to the two-dimensional case we can use a mask type $H_P(u) H_Q(v)$. But in this case we will emphasize the contour along the axes x and y . To avoid this, we can make a mask and transmission at each point of this mask is equal to the transmission in the opposite point, but with a phase difference πP . This mask has a transmittance of $\exp(iP\varphi)$ and hence is a spiral phase plate.

In [51] the authors used a liquid-crystal spatial light modulator (SLM) in place of the SPP. The light from an argon laser passed through a lens with a focal length of 36.8 cm and illuminated either a slit with a width of 200 mm or a circular aperture with a diameter of 300 mm. The SLM was placed in the Fourier plane of the lens, where the Fraunhofer diffraction pattern from the input field formed.

Figure 7.7 shows profiles of the diffraction pattern in the output plane of the Fourier correlator, when $P = 0$ (a), $P = 1/2$ (b), $P = 1$ (c), $P = 3/2$ (d).

Figure 7.8 shows the diffraction pattern in the output plane of the Fourier correlator when the depicted object is a circular aperture ($P = 0$ (a), $P = 1/2$ (b), $P = 1$ (c), $P = 3/2$ (d)).

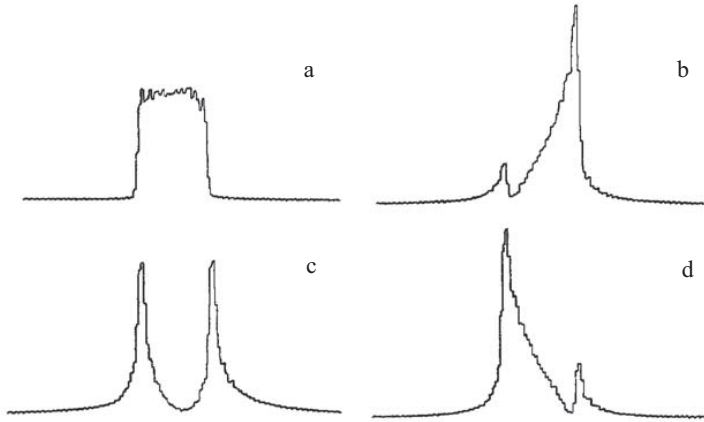


Fig. 7.7. Slit image (the mask was not used in the correlator), the results of the Hilbert transform of the order of $P = 1/2$ (a), $P = 1$ (b), $P = 3/2$ (c).

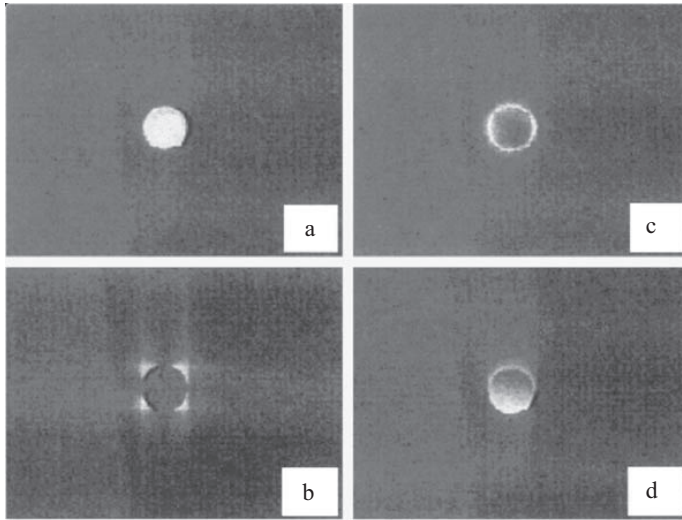


Fig. 7.8. Diffraction pattern in the output plane of the Fourier correlator in the absence of a mask (a), with a mask $H_1(u)H_1(v)$ (b), with a mask $\exp(i\varphi)$ (c) with a mask $\exp(i\varphi/2)$ (d).

7.2.3. Diffraction of a Gaussian beam on SPP: scalar theory

Fresnel diffraction of Gaussian beam on SPP

(7.18)

In [28, 52] explicit analytical expressions describing the Fresnel diffraction of Gaussian beam on the SPP were derived. At a distance z , the complex amplitude of the light field in the paraxial approximation has the form:

$$\begin{aligned}
E_n(\rho, \theta, z) &= \frac{(-i)k}{2\pi z} \int_0^\infty \int_0^{2\pi} E_n^0(r, \varphi) \exp\left\{\frac{ik}{2z} [r^2 + \rho^2 - 2r\rho \cos(\varphi - \theta)]\right\} r dr d\varphi = \\
&= \frac{(-i)^{n+1} \sqrt{\pi}}{2} \left(\frac{z_0}{z}\right)^2 \left(\frac{\rho}{w}\right) \left[1 + \left(\frac{z_0}{z}\right)^2\right]^{\frac{3}{4}} \exp\left[i\frac{3}{2} \arctg\left(\frac{z_0}{z}\right) + i\frac{k\rho^2}{2R(z)} - \frac{\rho^2}{w^2(z)} + in\theta\right] \times \\
&\times \left\{I_{\frac{n-1}{2}}\left[\rho^2\left(\frac{1}{w^2(z)} + \frac{ik}{2R_0(z)}\right)\right] - I_{\frac{n+1}{2}}\left[\rho^2\left(\frac{1}{w^2(z)} + \frac{ik}{2R_0(z)}\right)\right]\right\}, \quad (7.18)
\end{aligned}$$

where

$$\begin{aligned}
w^2(z) &= 2w^2 \left[1 + \left(\frac{z}{z_0}\right)^2\right], \quad R(z) = 2z \left[1 + \left(\frac{z_0}{z}\right)^2\right] \cdot \left[2 + \left(\frac{z_0}{z}\right)^2\right]^{-1}, \\
R_0(z) &= 2z \left[1 + \left(\frac{z}{z_0}\right)^2\right], \quad z_0 = \frac{kw^2}{2}, \quad E_n^0(r, \varphi) = \exp\left[-\left(\frac{r}{w}\right)^2 + in\varphi\right],
\end{aligned}$$

$I_\nu(x)$ is the Bessel function of second kind and the ν -th order.

In [52] by a limiting transition from diffraction in the Fresnel zone to the far field an expression was derived for the Fraunhofer diffraction of the Gaussian beam on the SPP.

When $z \gg z_0$

$$\begin{aligned}
w^2(z) &\approx 2w^2 \frac{z^2}{z_0^2}, \quad R(z) \approx z, \quad R_0(z) \approx \frac{2z^3}{z_0^2}. \\
E_n(\rho, \theta, z \rightarrow \infty) &= \frac{(-i)^{n+1} \sqrt{\pi}}{2} \left(\frac{z_0}{z}\right)^2 \left(\frac{\rho}{w}\right) \times \\
&\times \exp(in\theta) \exp\left[-\frac{\rho^2}{w^2(z)}\right] \left\{I_{\frac{n-1}{2}}\left(\frac{\rho^2}{w^2(z)}\right) - I_{\frac{n+1}{2}}\left(\frac{\rho^2}{w^2(z)}\right)\right\}. \quad (7.19)
\end{aligned}$$

(7.19) yields an expression for the intensity of the Gaussian beam with a phase singularity in the far diffraction zone

$$\begin{aligned}\hat{I}_n(\rho, z \rightarrow \infty) &\approx \frac{\pi}{4} \left(\frac{z_0}{z} \right)^4 \left(\frac{\rho}{w} \right)^2 \exp \left[-\frac{2\rho^2}{w^2(z)} \right] \left\{ I_{\frac{n-1}{2}} \left(\frac{\rho^2}{w^2(z)} \right) - I_{\frac{n+1}{2}} \left(\frac{\rho^2}{w^2(z)} \right) \right\}^2 = \\ &= \frac{\pi}{2} \left(\frac{z_0}{z} \right)^2 x \exp(-2x) \left\{ I_{\frac{n-1}{2}}(x) - I_{\frac{n+1}{2}}(x) \right\}^2, \quad (7.20)\end{aligned}$$

$$\text{where } x = \left(\frac{\rho z_0}{\sqrt{2} z w} \right)^2$$

Fraunhofer diffraction of a Gaussian beam on the SPP

Above we obtained explicit analytical expressions describing the Fresnel diffraction of the Gaussian beam on the SPP. By limiting transition from diffraction in the Fresnel zone to the far zone, an expression was also obtained for the Fraunhofer diffraction of a Gaussian beam on the SPP (the expression (7.20)). In this section, we derive analytical formulas to describe the Fraunhofer diffraction of the Gaussian beam on the SPP, located in its waist. The Fraunhofer diffraction pattern is formed in the focal plane of a spherical lens.

Consider the initial function in the form of:

$$f'_n(r, \theta) = \exp \left(-\frac{r^2}{w^2} + in\theta \right), \quad (7.21)$$

where w is the waist radius of the Gaussian beam. Then the complex amplitude of Fraunhofer diffraction of a Gaussian beam at the waist on the SPP will be described by the expression:

$$F'_n(\rho, \varphi) = \frac{(-i)^{n+1} k}{f} \exp(in\varphi) \int_0^\infty \exp \left(-\frac{r^2}{w^2} \right) J_n \left(\frac{k}{f} r \rho \right) r dr. \quad (7.22)$$

The known reference integral [53]:

$$\int_0^\infty \exp(-px^2) J_n(cx) x dx = \frac{c\sqrt{\pi}}{8p^{3/2}} \exp \left(-\frac{c^2}{8p} \right) \left[I_{(n-1)/2} \left(\frac{c^2}{8p} \right) - I_{(n+1)/2} \left(\frac{c^2}{8p} \right) \right], \quad (7.23)$$

where $I_\nu(x)$ is the modified Bessel function or the Bessel function of second kind. In view of (7.23) the expression (7.22) can be rewritten as:

$$F'_n(\rho, \varphi) = (-i)^{n+1} \exp(in\varphi) \left(\frac{kw^2}{4f} \right) \sqrt{2\pi x} \exp(-x) \left[I_{(n-1)/2}(x) - I_{(n+1)/2}(x) \right], \quad (7.24)$$

where

$$x = \frac{1}{2} \left(\frac{kw\rho}{2f} \right)^2.$$

The function of the intensity of the Fraunhofer diffraction pattern of the Gaussian beam on the SPP has the form:

$$\bar{I}'_n(\rho) = |F'_n(\rho, \varphi)|^2 = 2\pi \left(\frac{kw^2}{4f} \right)^2 x \exp(-2x) [I_{(n-1)/2}(x) - I_{(n+1)/2}(x)]^2. \quad (7.25)$$

From equation (7.25) we can see that for $x = 0$ at the centre of the Fourier plane intensity will be zero ($n \neq 0$): $\bar{I}'_n(0) = 0$. The factors $x \exp(-2x)$ in equation (7.25) show that an annular intensity distribution forms in the far zone. The radius of the ring can be found from the equation [28]:

$$(n - 4x)I_{(n-1)/2}(x) + (n + 4x)I_{(n+1)/2}(x) = 0. \quad (7.26)$$

We find the form of the function of the intensity on the outer side of the ring at $\rho \rightarrow \infty$ (or $x \rightarrow \infty$). For this we use the asymptotics of the Bessel function:

$$I_\nu(x) \approx \frac{\exp(x)}{\sqrt{2\pi x}} \left(1 - \frac{4\nu^2 - 1}{8x} \right), \quad x \gg 1. \quad (7.27)$$

Then instead of (7.25) at $x \rightarrow \infty$ we obtain:

$$\bar{I}'_n(\rho) \approx \left(\frac{nf}{k\rho^2} \right)^2. \quad (7.28)$$

It is interesting that equation (7.28) does not depend on the radius of the Gaussian beam waist. From this match we can conclude that the asymptotic behaviour of the intensity at $\rho \rightarrow \infty$ is determined only by the number of the SPP, the size of the focus of a spherical lens and the wavelength of the radiation and does not depend on the amplitude and phase parameters of the beam illuminating the SPP.

Note that the expression (7.28) can be obtained from equation (7.25), letting go to infinity the radius of the Gaussian beam $w \rightarrow \infty$ at a fixed ρ .

We find the form of the function of intensity inside the ring. When ρ tends to zero (for fixed w) the argument of the Bessel function x also tends to zero, and we can use the first terms of expansion of the cylindrical function into a series:

$$I_\nu(x) \approx \left(\frac{x}{2} \right)^\nu \Gamma^{-1}(\nu + 1), \quad x \ll 1, \quad (7.29)$$

where $\Gamma(x)$ is the gamma function. Then instead of (7.25) when we obtain:

$$\bar{I}'_n(\rho) \approx \pi \Gamma^{-2}\left(\frac{n+1}{2}\right) \left(\frac{kw^2}{f} \right) \left(\frac{kw\rho}{4f} \right)^{2n}. \quad (7.30)$$

From equation (7.30) we see that the intensity near the centre of the Fourier plane increases as the degree $2n$ of the radial coordinate:

$$\bar{I}'_n(\rho) \approx (w\rho)^{2n}, \quad \rho \ll 1. \quad (7.31)$$

If in addition to ρ tending to zero the Gaussian beam radius w should tend to infinity so that their product $w\rho$ remained constant, from equation (7.30) it follows that the intensity near the centre of the Fourier plane will tend to infinity as the square of the radius of the waist:

$$\bar{I}'_n(\rho \rightarrow 0, w \rightarrow \infty) \approx w^2, \quad \rho w = \text{const}, \quad (7.32)$$

but in the most central point at $\rho = 0$, the intensity will be zero $\bar{I}'_n(\rho = 0) = 0$, for any w .

For the experiments made with a 32-level SPP we generated a light field with the singularity of the second order. The size of the element is equal to $2.5 \times 2.5 \text{ mm}^2$, and the size of the frame $5 \times 5 \text{ mm}^2$. These SPPs were designed for the wavelength $\lambda = 633 \text{ nm}$. The depth of the microrelief, measured using a contact profilometer was 1320 nm . The optimum depth of the 32-level microrelief was 1341 nm on the assumption that the refractive index of the resist is $n_r = 1.457$ (exact value unknown). Thus, the deviation from the optimum depth is only about 1%. In [28] SPP was manufactured by the same technology, but for a wavelength of 514 nm , and the experiments were carried out at a wavelength of 543 nm . This led to the formation of a low quality tubular beam. The design of the SPP and the experiments were conducted using the same wavelength of a helium–neon laser at 633 nm . Therefore, the intensity distribution of the generated beam actually has a circular symmetry.

Figure 7.9a shows the estimated distribution of the phase (white colour indicates zero phase, black 2π ($1-1/N$), where N is the number of quantization levels). Figure 7.9b shows the microrelief of the SPP obtained using the interferometer NEWVIEW 5000 Zygo (200-fold magnification, italic type).

The annular intensity distribution in Fig. 7.10a was obtained as a result of diffraction of a Gaussian beam with a waist radius of $\sigma = 0.8 \text{ mm}$ on the SPP of the second order ($n = 2$). As a result of inexact matching the centre of the Gaussian beam and the centre of the SPP the circular symmetry on the diffraction pattern is violated.

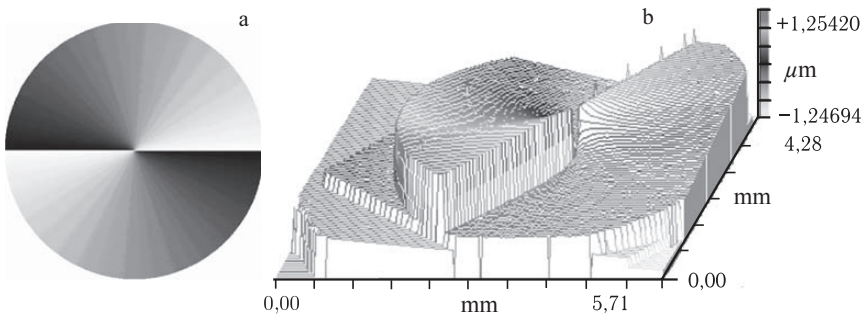


Fig. 7.9. The generation of the laser field with a phase singularity of the second order: (a) theoretical phase distribution, (b) the central part of the microrelief of the SPP.

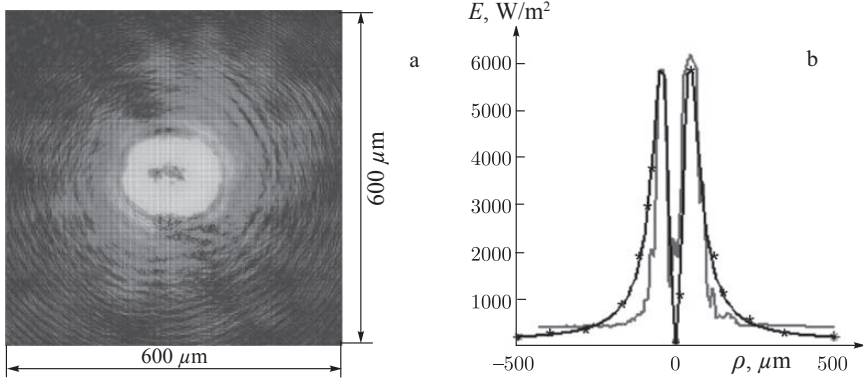


Fig. 7.10. Fraunhofer diffraction of the Gaussian beam (waist radius $\sigma = 0.1$ mm) on the SPP of the order $n = 2$; a) diffraction pattern; b) solid curve – experimental, and the curve with asterisks – theoretical intensity distribution.

Figure 7.10b shows a comparison of theoretical and experimental profiles of annular intensity distributions shown in Fig. 7.10a. The graph of the intensity in Fig. 7.10b is calibrated taking into account the power of the illuminating laser beam, measured by a wattmeter with an accuracy of 15%.

The radius of the ring in Fig. 7.10b can be obtained by using the ratio from [28]: $\rho_2 = 0.46 \lambda f / \sigma = 45 \mu\text{m}$. Figure 7.10b shows that the experimental and theoretical curves agree quite well.

7.2.4 Diffraction of a Gaussian beam on TFP: vector theory

This section analyzes the diffraction of Gaussian beam on TFP in the vector theory, and analytical expressions are derived for the longitudinal field component, which, as shown numerically, in some cases makes a significant contribution.

It is known that the propagation of light in free space is described by the Rayleigh–Sommerfeld diffraction integrals [54, 55]:

$$\begin{cases} E_x(u, v, z) = -\frac{1}{2\pi} \iint_{R^2} E_x(x, y, 0) \frac{\partial}{\partial z} \left[\frac{\exp(ikR)}{R} \right] dx dy, \\ E_y(u, v, z) = -\frac{1}{2\pi} \iint_{R^2} E_y(x, y, 0) \frac{\partial}{\partial z} \left[\frac{\exp(ikR)}{R} \right] dx dy, \\ E_z(u, v, z) = \frac{1}{2\pi} \iint_{R^2} \left\{ E_x(x, y, 0) \frac{\partial}{\partial x} \left[\frac{\exp(ikR)}{R} \right] + E_x(u, v, 0) \frac{\partial}{\partial y} \left[\frac{\exp(ikR)}{R} \right] \right\} dx dy, \end{cases} \quad (7.33)$$

where $R = [(u-x)^2 + (v-y)^2 + z^2]^{1/2}$, (x, y) are Cartesian coordinates in the CPP plane $z = 0$, (u, v) are the Cartesian coordinates in the plane, at a distance z from the plane of the CPP, $k = 2\pi/\lambda$ is the wave number.

In the calculation of these integrals the factors containing derivatives of the function $R^{-1} \exp(ikR)$ are normally replaced by approximate expressions. In the paraxial approximation this is done as follows: the following change is made in the exponent of rapidly oscillating exponentials

$$R \approx z + \frac{(u-x)^2 + (v-y)^2}{2z}, \quad (7.34)$$

but in other cases it is assumed that $R \approx z$. After these transformations, instead of (7.33) we obtain the following expressions:

$$\begin{cases} E_{x,y}(u, v, z) \approx -\frac{ik}{2\pi z} \exp(ikz) \iint_{R^2} E_{x,y}(x, y, 0) \exp\left\{\frac{ik}{2z}[(u-x)^2 + (v-y)^2]\right\} dx dy, \\ E_z(u, v, z) \approx \frac{ik}{2\pi z^2} \exp(ikz) \iint_{R^2} [(x-u)E_x(x, y, 0) + (y-v)E_y(x, y, 0)] \times \\ \times \exp\left\{\frac{ik}{2z}[(u-x)^2 + (v-y)^2]\right\} dx dy, \end{cases} \quad (7.35)$$

where $E_{x,y}$ is either E_x or E_y . From (7.35) it can be seen that for the transverse components we obtain the well-known Fresnel transformation. In [56] the authors used a less rough approximation: the following change is made in the exponent of rapidly oscillating exponentials

$$R \approx \sqrt{u^2 + v^2 + z^2} + \frac{(u-x)^2 + (v-y)^2}{2\sqrt{u^2 + v^2 + z^2}}, \quad (7.36)$$

but in other cases it is considered that $R \approx (u^2 + v^2 + z^2)^{1/2}$. After these transformations, instead of (7.33) we can write approximately:

$$\left\{ \begin{aligned} E_{x,y}(u,v,z) &\approx -\frac{ikz \exp\left(ik\sqrt{u^2+v^2+z^2}\right)}{2\pi(u^2+v^2+z^2)} \times \\ &\times \iint_{R^2} E_{x,y}(x,y,0) \exp\left[\frac{ik}{2\sqrt{u^2+v^2+z^2}}(x^2+y^2-2ux-2vy)\right] dx dy, \\ E_z(u,v,z) &\approx \frac{ik}{2\pi(u^2+v^2+z^2)} \exp\left(ik\sqrt{u^2+v^2+z^2}\right) \times \\ &\times \iint_{R^2} [(x-u)E_x(x,y,0) + (y-v)E_y(x,y,0)] \\ &\times \exp\left[\frac{ik}{2\sqrt{u^2+v^2+z^2}}(x^2+y^2)\right] \exp\left[-\frac{ik}{\sqrt{u^2+v^2+z^2}}(ux+vy)\right] dx dy. \end{aligned} \right. \quad (7.37)$$

We can see that in formula $(u^2+v^2)^{1/2} \ll z$ (7.37) becomes (7.35).

In the case when the beam in the input plane has a vortical component, i.e.

$$\begin{cases} E_x(r,\varphi,0) \equiv A_x(r) \exp(in\varphi), \\ E_y(r,\varphi,0) \equiv A_y(r) \exp(in\varphi), \end{cases} \quad (7.38)$$

where (r, φ) are the polar coordinates in the plane $z = 0$, the double integrals in (7.35) and (7.37) after transition to the polar coordinates can be reduced to single.

In the case of the paraxial approximation (7.35) we obtain the following expression:

$$\left\{ \begin{aligned} E_{x,y}(\rho,\theta,z) &= (-i)^{n+1} \frac{k}{z} \exp\left(\frac{ik\rho^2}{2z} + in\theta + ikz\right) \int_0^\infty A_{x,y}(r) \exp\left(\frac{ikr^2}{2z}\right) J_n\left(\frac{k\rho r}{z}\right) r dr, \\ E_z(\rho,\theta,z) &= (-i)^n \frac{k}{z^2} \exp\left(\frac{ik\rho^2}{2z} + in\theta + ikz\right) \times \\ &\times \left[\exp(i\theta) \int_0^\infty \frac{A_x(r) - iA_y(r)}{2} \exp\left(\frac{ikr^2}{2z}\right) J_{n+1}\left(\frac{k\rho r}{z}\right) r^2 dr - \right. \\ &- \exp(-i\theta) \int_0^\infty \frac{A_x(r) + iA_y(r)}{2} \exp\left(\frac{ikr^2}{2z}\right) J_{n-1}\left(\frac{k\rho r}{z}\right) r^2 dr - \\ &\left. - i\rho \int_0^\infty [A_x(r) \cos\theta + A_y(r) \sin\theta] \exp\left(\frac{ikr^2}{2z}\right) J_n\left(\frac{k\rho r}{z}\right) r dr \right], \end{aligned} \right. \quad (7.39)$$

where (ρ, θ) are the polar coordinates in the plane at distance z from the plane of the SPP; $J_n(x)$ is the Bessel function of n -th order.

In the case of non-paraxial approximation (7.37) we obtain expressions similar in form but more accurate:

$$\left\{ \begin{aligned} E_{x,y}(\rho, \theta, z) &= (-i)^{n+1} \frac{kz \exp\left(in\theta + ik\sqrt{\rho^2 + z^2}\right)}{\rho^2 + z^2} \times \\ &\quad \times \int_0^\infty A_{x,y}(r) \exp\left(\frac{ikr^2}{2\sqrt{\rho^2 + z^2}}\right) J_n\left(\frac{k\rho r}{\sqrt{\rho^2 + z^2}}\right) r dr, \\ E_z(\rho, \theta, z) &= (-i)^n \frac{k}{\rho^2 + z^2} \exp\left(ik\sqrt{\rho^2 + z^2} + in\theta\right) \times \\ &\quad \times \left[\exp(i\theta) \int_0^\infty \frac{A_x(r) - iA_y(r)}{2} \exp\left(\frac{ikr^2}{2\sqrt{\rho^2 + z^2}}\right) J_{n+1}\left(\frac{k\rho r}{\sqrt{\rho^2 + z^2}}\right) r^2 dr - \right. \\ &\quad - \exp(-i\theta) \int_0^\infty \frac{A_x(r) + iA_y(r)}{2} \exp\left(\frac{ikr^2}{2\sqrt{\rho^2 + z^2}}\right) J_{n-1}\left(\frac{k\rho r}{\sqrt{\rho^2 + z^2}}\right) r^2 dr - \\ &\quad \left. - i\rho \int_0^\infty [A_x(r) \cos\theta + A_y(r) \sin\theta] \exp\left(\frac{ikr^2}{2\sqrt{\rho^2 + z^2}}\right) J_n\left(\frac{k\rho r}{\sqrt{\rho^2 + z^2}}\right) r dr \right]. \end{aligned} \right. \quad (7.40)$$

If a Gaussian beam falls on the SPF, i.e. $A_{x,y}(r) = B_{x,y} \exp(-r^2/w^2)$, the integrals in (7.39) and (7.40) can be calculated using the following reference integral [53]:

$$\int_0^\infty \exp(-px^2) J_\nu(cx) x dx = \frac{c\sqrt{\pi}}{8p^{3/2}} \exp(-y) \left[I_{\frac{\nu-1}{2}}(y) - I_{\frac{\nu+1}{2}}(y) \right], \text{Re } \nu > -2, \quad (7.41)$$

where $y = c^2/(8p)$, $I_n(x)$ is the Bessel function of second kind, and with the help of another integral, which can be obtained from (7.41):

$$\begin{aligned} \int_0^\infty \exp(-px^2) J_\nu(cx) x^2 dx &= \frac{\sqrt{\pi}}{8p^{3/2}} \exp(-y) \times \\ &\quad \times \left\{ (\nu + 2 - 3y) \left[I_{\frac{\nu}{2}}(y) - I_{\frac{\nu+2}{2}}(y) \right] + y \left[I_{\frac{\nu-2}{2}}(y) - I_{\frac{\nu+4}{2}}(y) \right] \right\}. \end{aligned} \quad (7.42)$$

After application of the integrals (7.41) and (7.42) to the expressions (7.40), we get:

$$\begin{cases}
E_{x,y}(\rho, \theta, z) = (-i)^{n+1} \frac{B_{x,y} k z \exp\left(in\theta + ik\sqrt{\rho^2 + z^2}\right)}{\rho^2 + z^2} \frac{c\sqrt{\pi}}{8p^{3/2}} \exp(-y) \left[I_{\frac{n-1}{2}}(y) - I_{\frac{n+1}{2}}(y) \right], \\
E_z(\rho, \theta, z) = (-i)^n \frac{k}{\rho^2 + z^2} \exp\left(ik\sqrt{\rho^2 + z^2} + in\theta\right) \frac{\sqrt{\pi}}{8p^{3/2}} \exp(-y) \times \\
\quad \times \left(\frac{B_x - iB_y}{2} \exp(i\theta) \left\{ (n+3-3y) \left[I_{\frac{n+1}{2}}(y) - I_{\frac{n+3}{2}}(y) \right] + y \left[I_{\frac{n-1}{2}}(y) - I_{\frac{n+5}{2}}(y) \right] \right\} - \right. \\
\quad \left. - \frac{B_x + iB_y}{2} \exp(-i\theta) \left\{ (n+1-3y) \left[I_{\frac{n-1}{2}}(y) - I_{\frac{n+1}{2}}(y) \right] + y \left[I_{\frac{n-3}{2}}(y) - I_{\frac{n+3}{2}}(y) \right] \right\} - \right. \\
\quad \left. - i(B_x \cos \theta + B_y \sin \theta) c \rho \left[I_{\frac{n-1}{2}}(y) - I_{\frac{n+1}{2}}(y) \right] \right).
\end{cases} \quad (7.43)$$

In equation (7.43) the notation:

$$p = \frac{1}{w^2} - \frac{ik}{2\sqrt{\rho^2 + z^2}}, \quad c = \frac{k\rho}{\sqrt{\rho^2 + z^2}}, \quad y = \frac{c^2}{8p}. \quad (7.44)$$

The Cartesian components of the vector of the strength of the electric field (7.43) in cylindrical coordinates describe the non-paraxial Gaussian beam diffraction on the SPP with a topological charge n . Note that for $B_z = \pm iB_y$ a Gaussian beam has a circular polarization, and with $B_x \neq 0, B_y = 0$ – linear polarization.

In the numerical simulation the integrals (7.39) and (7.40) are calculated by the method of rectangles and compared with the values obtained with the formulas (7.43) and the formulas obtained from (7.43) for the paraxial approximation. Thus, the resulting expression (7.42) was verified.

This was followed by numerical comparison of the paraxial approximation (7.39) and the more accurate non-paraxial approximation (7.40). The simulation results are shown in Fig. 7.11. We used the following parameters: wavelength $\lambda = 633$ nm, the radius of the Gaussian beam waist $w = 1 \mu\text{m}$, the order of SPP $n = 3$, and the distance along the optical axis $z = 10$ mm, the amplitudes of the Gaussian beam $B_x = 1$ and $B_y = 0.2i$ (elliptical polarization).

Figure 7.11 shows that the transverse components of the vector of the strength of the electric field obtained with the paraxial and non-paraxial approximations differ from each other (the maximum error was 14%). The longitudinal component in this case is small.

Figure 7.12 shows the diffraction of the same Gaussian beam, but at a distance $z = 10 \mu\text{m}$.

Figure 7.12 shows that under these conditions it is already important to consider the effect on the intensity of the longitudinal projection of the vector of the electric field, as it is about 3% of the transverse projection.

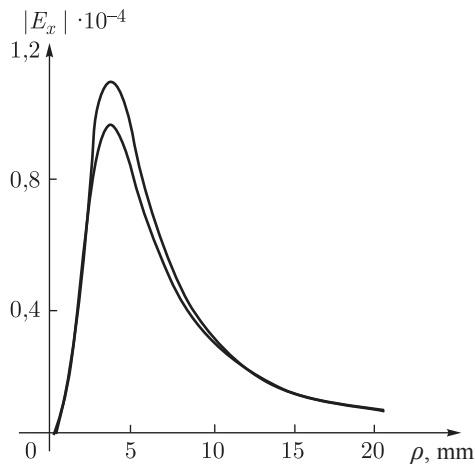


Fig. 7.11. Diffraction of Gaussian beam on SPP: the transverse component $|E_x|$ (a) and longitudinal component $|E_z|$ (b) (solid line – in the paraxial approximation, dashed – in non-paraxial approximation).

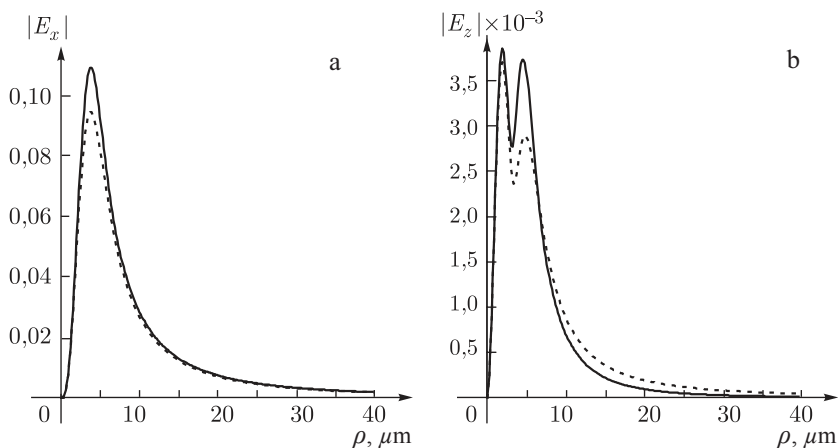


Fig. 7.12. Diffraction of a Gaussian beam on the SPP at $z = 10 \mu\text{m}$: the transverse component $|E_x|$ (a) and the longitudinal component $|E_z|$ (b) (solid line – in the paraxial approximation, dashed - in non-paraxial approximation).

Figure 7.13 shows the radial distribution of the modulus of the electric vector obtained by the formula (7.43) (solid line) and using the diffraction integral Rayleigh-Sommerfeld (7.33) (mean computational complexity of the individual values are shown by dots). Calculation parameters are the same as that for Fig. 7.11 and 7.12. The distance was taken equal to $z = 10 \text{ mm}$.

Figure 7.13 shows that the formula (7.43) yields results virtually identical with the exact formula (7.33).

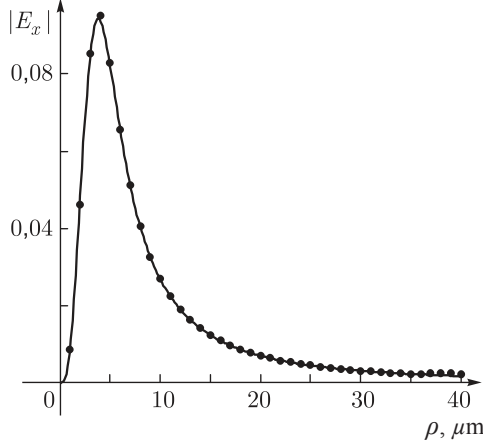


Fig. 7.13. Comparison of the calculation results for the diffraction pattern of the Gaussian beam, using the approximate formula (7.43) (solid line) and the exact formula (7.33) (graph is shown in dots).

7.2.5. Fresnel diffraction of a restricted plane wave on SPP

Consider the Fresnel diffraction of a restricted plane wave on SPP. Paraxial wave diffraction on the SPP will be described by the following transformation (derived from the Fresnel transform):

$$\begin{aligned}
 E_n(\rho, \theta, z) &= \frac{(-i)^{n+1} k}{z} \exp\left(\frac{ik\rho^2}{2z} + in\theta\right) \int_0^R \exp\left(\frac{ikr^2}{2z}\right) J_n\left(\frac{k}{z} r \rho\right) r dr = \\
 &= \exp\left(\frac{iz_0 \bar{\rho}^2}{z} + in\theta\right) \frac{2\left(\frac{-iz_0}{z}\right)^{n+1} \bar{\rho}^n}{n!} \sum_{m=0}^{\infty} \frac{\left(\frac{iz_0}{z}\right)^m}{(2m+n+2)m!} {}_1F_2\left[\frac{2m+n+2}{2}, \frac{2m+n+4}{2}, n+1; -\left(\frac{z_0 \bar{\rho}}{z}\right)^2\right],
 \end{aligned} \tag{7.45}$$

where $z_0 = kR^2/2$ is the Rayleigh length, $\bar{\rho} = \rho/R$. Equation (7.45) shows that at $n \neq 0$ in the centre of the beam at $\rho = 0$ the amplitude is zero $E_n(\rho = 0, \theta, z)$ for all z , except $z = 0$. Equation (7.45) also shows that with increasing z in the series of hypergeometric functions contributions are provided only the first few terms in the series, and if $z \rightarrow \infty$ ($z \gg z_0$, far field) the contribution to the amplitude will come only from the first term at $m = 0$. Note that in (7.45) the integer part of ratio z_0/z is equal to the Fresnel number. Note also that the expression (7.45) at $n = 0$ (no SPP) describes the Fresnel diffraction of a plane wave on a circular aperture of radius R :

$$E_0(\rho, z) = (-1) \exp\left(\frac{iz_0 \bar{\rho}^2}{z}\right) \sum_{m=0}^{\infty} \frac{\left(\frac{iz_0}{z}\right)^{m+1}}{(m+1)!} {}_1F_2\left[m+1, m+2, 1; -\left(\frac{z_0 \bar{\rho}}{z}\right)^2\right]. \tag{7.46}$$

From (7.46) can be a simple dependence of the complex amplitude of the light field on the optical axis ($\rho = 0$) of the distance z to the diaphragm:

$$E_0(\rho = 0, z) = 1 - \exp\left(\frac{iz_0}{z}\right). \quad (7.47)$$

The expression (7.47) coincides with that obtained previously [57].

Figure 7.14 shows the results of the comparison of experiment and calculation. In Fig. 7.14 shows a surface profile of the SPP with the number $n = 3$ and a diameter of 2.5 mm, visualized with an interferometer Newview 5000 Zygo (increase by 200 times). The SPT profile differs from the ideal of 4.3%, while the SPP itself has 32 gradations of relief and was produced by a low-contrast negative resist XAR-N7220 by direct write electron beam with the lithographer Leica LION LV1 with a resolution of 5 microns.

Fig. 7.14 b, c shows the experimental and calculated diffraction pattern of a plane wave at the SPP with a radius $R = 1.25$ mm and a wavelength $\lambda = 0.633$ mm at a distance $z = 80$ mm. Both diffraction patterns have the same number of rings (8 rings).

Figure 7.15 shows the result of registering with the CCD-camera pictures of the Fraunhofer diffraction at the lens focus ($f = 150$ mm) obtained for a plane wave with a radius of 1.25 mm, 0.633 mm wavelength and TFP $n = 3$.

The relative standard deviation of the theoretical and experimental curves in Fig. 7.15 b, was 14.3%.

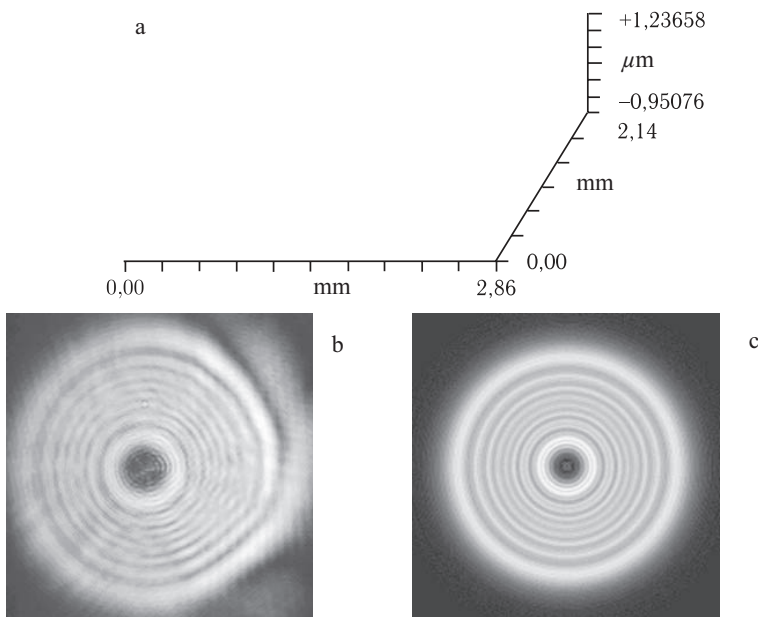


Fig. 7.14. Profile of the surface of SPP ($n = 3$) (a), Fresnel diffraction pattern of a plane wave with radius $R = 1.25$ mm and wavelength $\lambda = 0.633$ μm at distance $z = 80$ mm from the SPP: experiment (b) and theory (c).

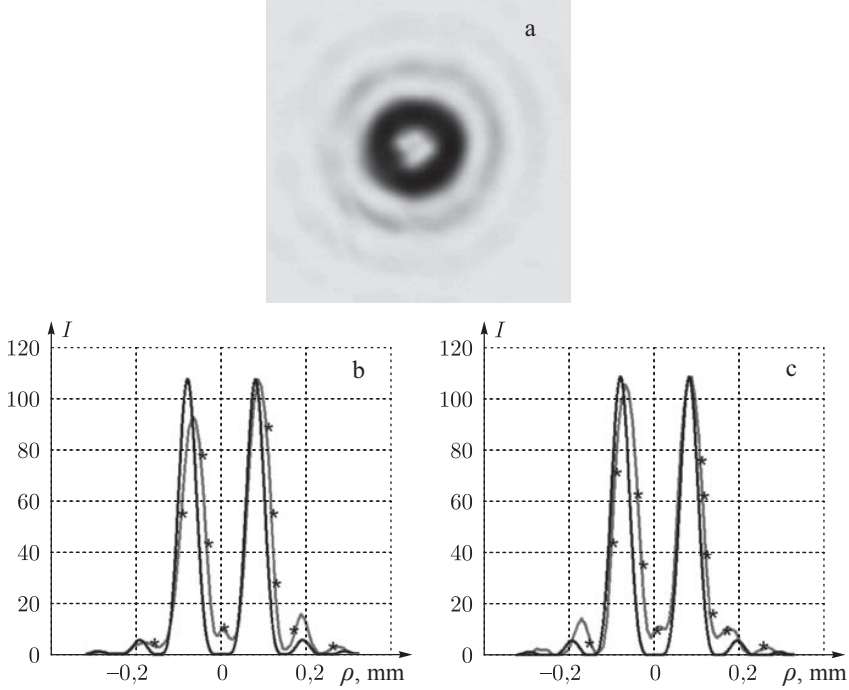


Figure 7.15. Fraunhofer diffraction pattern (negative) on SPP with the number $n = 3$, a plane wave with a radius of 1.25 mm and wavelength $0.633 \mu\text{m}$, formed in the focal plane of Fourier lens with a focal length of 150 mm: the intensity distribution of (negative) (a), vertical (b) and horizontal (c) in the intensity section (solid curves – theory, * – * – experiment).

In this section we derived analytical expressions that describe the paraxial diffraction of a restricted plane wave on the SPP. Using a SPP produced with high accuracy with the number $n = 3$ we obtained Fresnel and Fraunhofer experimental diffraction patterns. Theory and experiment are consistent with an average error of not more than 15%.

7.2.6. Diffraction of a restricted plane wave on TFP: paraxial vectorial theory

From the expression (7.39) it follows that in the paraxial approximation, the expressions of the electromagnetic field component have the following form:

$$E_x(\rho, \theta, z) = (-i)^{n+1} \frac{k}{z} \exp\left(\frac{ik\rho^2}{2z} + in\theta + ikz\right) \int_0^\infty A_x(r) \exp\left(\frac{ikr^2}{2z}\right) J_n\left(\frac{k\rho r}{z}\right) r dr, \quad (7.48)$$

$$E_y(\rho, \theta, z) = (-i)^{n+1} \frac{k}{z} \exp\left(\frac{ik\rho^2}{2z} + in\theta + ikz\right) \int_0^\infty A_y(r) \exp\left(\frac{ikr^2}{2z}\right) J_n\left(\frac{k\rho r}{z}\right) r dr, \quad (7.49)$$

$$\begin{aligned} E_z(\rho, \theta, z) = & \frac{(-i)^n k}{2z^2} \exp\left(\frac{ik\rho^2}{2z} + in\theta + ikz\right) \times \\ & \times \left\{ \exp(i\theta) \int_0^\infty [A_x(r) - iA_y(r)] \exp\left(\frac{ikr^2}{2z}\right) J_{n+1}\left(\frac{k\rho r}{z}\right) r^2 dr - \right. \\ & - \exp(-i\theta) \int_0^\infty [A_x(r) + iA_y(r)] \exp\left(\frac{ikr^2}{2z}\right) J_{n-1}\left(\frac{k\rho r}{z}\right) r^2 dr - \\ & \left. 2i\rho \int_0^\infty [A_x(r) \cos\theta + A_y(r) \sin\theta] \exp\left(\frac{ikr^2}{2z}\right) J_n\left(\frac{k\rho r}{z}\right) r dr \right\}, \end{aligned} \quad (7.50)$$

where $J_n(x)$ is the Bessel function of n -th order.

In the case where in the plane $z = 0$ there is a spiral phase plate (SPP) of radius R and n -th order, and a lens with a focal length f , we obtain the expression:

$$A_x(r) \equiv A_x \text{circ}\left(\frac{r}{R}\right) \exp\left(-\frac{ikr^2}{2f}\right), \quad (7.51)$$

$$A_y(r) \equiv A_y \text{circ}\left(\frac{r}{R}\right) \exp\left(-\frac{ikr^2}{2f}\right), \quad (7.52)$$

where A_x and A_y are the complex amplitudes of a plane wave incident on the SPP with a lens. Then, at a distance z an electromagnetic field with the following components ($E_{x,y}$ – it is either E_{x^*} or E_y) will form:

$$E_{x,y}(\rho, \theta, z) = (-i)^{n+1} \frac{kA_{x,y}}{z} \exp\left(\frac{ik\rho^2}{2z} + in\theta + ikz\right) \int_0^R \exp\left[\frac{ikr^2}{2}\left(\frac{1}{z} - \frac{1}{f}\right)\right] J_n\left(\frac{k\rho r}{z}\right) r dr, \quad (7.53)$$

$$\begin{aligned} E_z(\rho, \theta, z) = & (-i)^n \frac{k}{2z^2} \exp\left(\frac{ik\rho^2}{2z} + in\theta + ikz\right) \times \\ & \times \left\{ (A_x - iA_y) \exp(i\theta) \int_0^R \exp\left[\frac{ikr^2}{2}\left(\frac{1}{z} - \frac{1}{f}\right)\right] J_{n+1}\left(\frac{k\rho r}{z}\right) r^2 dr - \right. \\ & - (A_x + iA_y) \exp(-i\theta) \int_0^R \exp\left[\frac{ikr^2}{2}\left(\frac{1}{z} - \frac{1}{f}\right)\right] J_{n-1}\left(\frac{k\rho r}{z}\right) r^2 dr - \end{aligned} \quad (7.54)$$

$$-2i\rho(A_x \cos \theta + A_y \sin \theta) \int_0^R \exp \left[\frac{ikr^2}{2} \left(\frac{1}{z} - \frac{1}{f} \right) \right] J_n \left(\frac{k\rho r}{z} \right) r dr \Bigg\}.$$

In the geometric focus of the lens, i.e. at $z = f$, the expressions can be simplified [58]:

$$\begin{aligned} E_{x,y}(\rho, \theta, z = f) &= (-i)^{n+1} \frac{kA_{x,y}}{f} \exp \left(\frac{ik\rho^2}{2f} + in\theta + ikf \right) \int_0^R J_n \left(\frac{k\rho r}{f} \right) r dr = \\ &= (-i)^{n+1} \frac{kA_{x,y}}{f} \exp \left(\frac{ik\rho^2}{2f} + in\theta + ikf \right) \times \\ &\quad \times \begin{cases} n \left[1 - J_0(y) - 2 \sum_{m=1}^{n/2-1} J_{2m}(y) \right] - yJ_{n-1}(y), n = 2p, \\ n \left[\int_0^y J_0(t) dt - 2 \sum_{m=1}^{(n-1)/2} J_{2m-1}(y) \right] - yJ_{n-1}(y), n = 2p+1, \end{cases} \end{aligned} \quad (7.55)$$

where $y = kR\rho/f$.

$$\begin{aligned} E_z(\rho, \theta, z = f) &= \frac{(-i)^n k}{2f^2} \exp \left(\frac{ik\rho^2}{2f} + in\theta + ikf \right) \times \\ &\quad \left\{ (A_x - iA_y) \exp(i\theta) \int_0^R J_{n+1} \left(\frac{k\rho r}{f} \right) r^2 dr - \right. \\ &\quad \left. - (A_x + iA_y) \exp(-i\theta) \int_0^R J_{n-1} \left(\frac{k\rho r}{f} \right) r^2 dr - \right. \\ &\quad \left. - 2i\rho(A_x \cos \theta + A_y \sin \theta) \int_0^R J_n \left(\frac{k\rho r}{f} \right) r dr \right\}. \end{aligned} \quad (7.56)$$

The last integral in (7.56) is calculated as in (7.55). For the first two integrals we can also obtain analytical expressions for even values of the order of SPP n , so as for $p = N \pm 1$

$$\int x^2 J_p(cx) dx = \frac{1-p^2}{c^3} \left[J_0(cx) + 2 \sum_{q=1}^{(p-3)/2} J_{2q}(cx) \right] - \frac{x^2}{c} J_{p-1}(x) - \frac{p+1}{c^2} x J_{p-2}(x). \quad (7.57)$$

For small orders of SPPP we obtain simple formulas. In particular, for $n = 2$:

$$E_{x,y}(\rho, \theta, z = f) = -\frac{2ifA_{x,y}}{k\rho^2} \exp\left(\frac{ik\rho^2}{2f} + i2\theta + ikf\right) \left[J_0\left(\frac{kR\rho}{f}\right) + \frac{kR\rho}{2f} J_1\left(\frac{kR\rho}{f}\right) - 1 \right], \quad (7.58)$$

$$\begin{aligned} E_z(\rho, \theta, z = f) = & -\frac{1}{2k\rho} \exp\left(\frac{ik\rho^2}{2f} + i2\theta + ikf\right) \times \\ & \left(-\frac{2kR^2}{f} (A_x \cos \theta + A_y \sin \theta) J_2\left(\frac{kR\rho}{f}\right) + \right. \\ & \left. + \left[\frac{4}{\rho} (A_x - iA_y) \exp(i\theta) - \frac{2ik\rho}{f} (A_x \cos \theta + A_y \sin \theta) \right] \times \right. \\ & \left. \times \left\{ \frac{f}{k\rho} \left[2 - 2J_0\left(\frac{kR\rho}{f}\right) \right] - RJ_1\left(\frac{kR\rho}{f}\right) \right\} \right). \end{aligned} \quad (7.59)$$

Let us consider two special cases of the circular polarization of the field in the initial plane.

At $A_y = -iA_x$:

$$\begin{aligned} E_z(\rho, \theta, z) = & \frac{iA_x R}{f\rho} \exp\left(\frac{ik\rho^2}{2f} + i\theta + ikf\right) \times \\ & \left\{ \frac{f}{kR} \left[2 - 2J_0\left(\frac{kR\rho}{f}\right) \right] - \rho J_1\left(\frac{kR\rho}{f}\right) - iR J_2\left(\frac{kR\rho}{f}\right) \right\}. \end{aligned} \quad (7.60)$$

At $A_y = iA_x$:

$$\begin{aligned} E_z(\rho, \theta, z) = & \frac{iA_x R}{f\rho} \exp\left(\frac{ik\rho^2}{2f} + i3\theta + ikf\right) \times \\ & \left(-iR J_2\left(\frac{kR\rho}{f}\right) + \left(1 - \frac{4f}{ik\rho^2} \right) \left\{ \frac{f}{kR} \left[2 - 2J_0\left(\frac{kR\rho}{f}\right) \right] - \rho J_1\left(\frac{kR\rho}{f}\right) \right\} \right). \end{aligned} \quad (7.61)$$

Figure 7.16 shows the distribution of the amplitude of the z -component of the electromagnetic field along the optical axis. Calculation parameters: wavelength $\lambda = 514.5$ nm, the aperture radius: $R = 2$ mm, the order of the SPP: $n = 1$.

Figures 7.17 and 7.18 show the amplitude distribution of the x - and z -components of the electromagnetic field along the radial coordinate. Calculation parameters: wavelength: $\lambda = 514.5$ nm, focal length of the lens: $f = 500$ mm, the order of the SPP: $n = 1$.

Figure 7.18 shows that the z -component of the amplitude can be several percent, so in some cases it makes sense to consider its existence, even in the paraxial case.

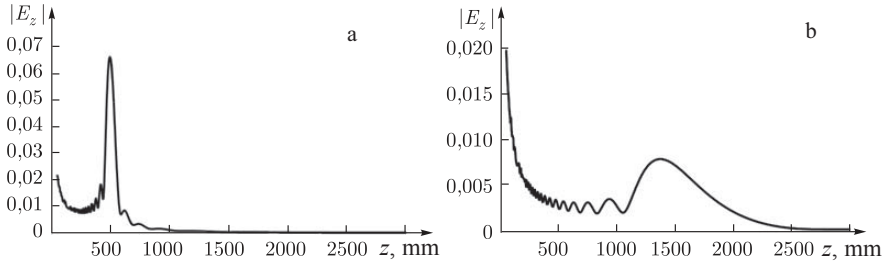


Figure 7.16. Absolute value of the z -component on the optical axis, calculated for $f = 500$ mm (a) and $f = 1500$ mm (b).

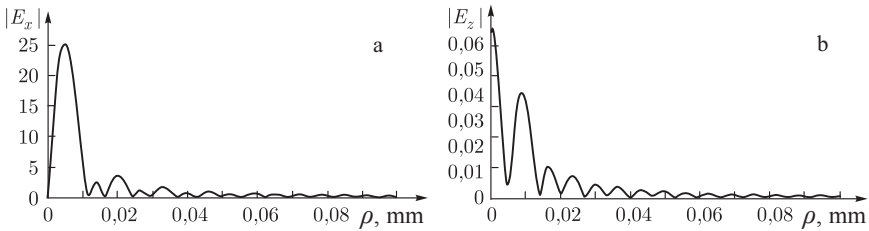


Fig. 7.17. The value of the amplitude in the plane $z = 500$ mm at an aperture radius of 2 mm. x -component (a) and z -component (b).

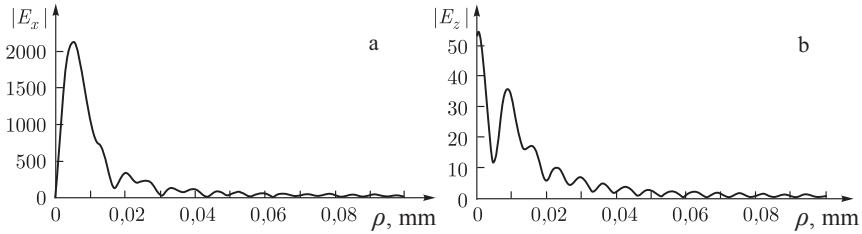


Fig. 7.18. The value of the amplitude in the plane $z = 500$ mm at an aperture radius of 20 mm. x -component (a) and z -component (b).

7.3. Quantized SPP with a restricted aperture, illuminated by a plane wave

There are many ways of making the SPP, such as multistage etching of silicon [59] or by ablation of polyamide substrates using an excimer laser [60]. The microrelief of the resultant SPP is stepped or quantized.

Multilevel SPPs were studied in [31, 61]. In [31] the efficiency of conversion of a Gaussian beam to the Laguerre–Gaussian mode (0,1) was theoretically calculated, and experiments were also carried out with a 16-level SPP produced by photolithography.

In [61] the authors found theoretically the minimum number of levels of the SPP phase (for the numbers $n < 8$), in which the finite-level SPPs slightly differ from the continuous SPPs. With the help of the finite-level SPP, produced on the basis of a liquid crystal cell, vortex laser beams with indices of singularity to 6 were formed in [61].

In [62, 63] attention was paid to the achromatic TFP, which forms almost the same eddy fields, if the wavelength of the illuminating radiation varies in a relatively wide range (140 nm). In these studies [31, 61–63] the SPP analyzed using a series expansion of angular harmonics:

$$\exp \left[i \operatorname{mod} \left(\frac{P\varphi}{2\pi} \right) \frac{2\pi n}{P} \right] = \sum_{m=-\infty}^{\infty} C_m \exp(im\varphi), \quad (7.62)$$

where $\operatorname{mod}(\dots)$ is an integer, P is the total number of phase levels of SPP, φ is the azimuthal angle of the polar coordinate system, n is the number of SPP, C_m are complex coefficients, $\exp(im\varphi)$ are the angular harmonics describing the transmission of the continuous SPP with the number m .

In this section, the finite-level SPP, bounded by a polygonal aperture (i.e., having the shape of the polygon) is considered. Moreover, the number of quantization levels of the SPP phase equals the number of sides of a regular polygon, bounding the aperture of the SPP. In this case it was possible to obtain analytical expressions as a finite sum of plane waves for the complex amplitude, which describes the Fraunhofer diffraction of a plane wave on a finite-level SPP, bounded by a regular polygon.

Note that the possibility of the formation of eddy fields using non-spiral phase plates was already considered [64]. In our case, unlike in [64], with an increase in the number of phase quantization levels (or the number of sides), the diffraction pattern in the far field tends to the diffraction pattern formed by a continuous SPP with a circular aperture.

The equation of the polygonal aperture

Let Ω be the polygon defined by the coordinates of its vertices $A_p(x_p, y_p)$, $p = \overline{0, P-1}$ where P is the number of vertices (see Fig. 7.19).

Let the equation of the polygon connecting the p -th and $(p+1)$ -th vertex is given by:

$$y = a_p x + b_p. \quad (7.63)$$

Let $f(x, y)$ be a function of two variables defined in R^2 as follows:

$$f(x, y) = \begin{cases} 1, & (x, y) \in \Omega, \\ 0, & (x, y) \notin \Omega. \end{cases} \quad (7.64)$$

It is known that the Fourier transform of such function $f(x, y)$ is calculated using the equation of the polygonal aperture [65]:

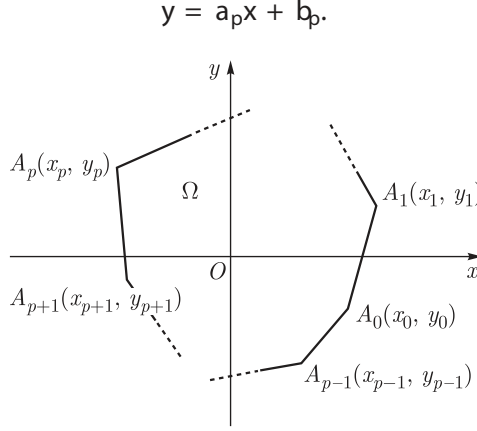


Fig. 7.19. DOE with a polygonal aperture.

$$\begin{aligned} \iint_{\Omega} \exp[\pm i(x\xi + y\eta)] dx dy &= \sum_{p=1}^P \frac{a_p - a_{p-1}}{(\xi + \eta a_{p-1})(\xi + \eta a_p)} \exp[\pm i(\xi x_p + \eta y_p)] = \\ &= \sum_{p=1}^P \frac{(y_{p+1} - y_p)(x_p - x_{p-1}) - (y_p - y_{p-1})(x_{p+1} - x_p)}{[\xi(x_{p+1} - x_p) + \eta(y_{p+1} - y_p)][\xi(x_p - x_{p-1}) + \eta(y_p - y_{p-1})]} \exp[\pm i(\xi x_p + \eta y_p)], \end{aligned} \quad (7.65)$$

where p refers to value of the modes (p, P) , i.e. $(x_p, y_p) = (x_0, y_0)$, $(x_{-1}, y_{-1}) = (x_{P-1}, y_{P-1})$, etc.

Then the complex amplitude describing Fraunhofer diffraction at polygonal apertures (Fig.7.19) of a plane wavelength λ at a focal length spherical lens is equal to f , is given by:

$$E(\xi, \eta) = -\frac{if}{2\pi k} \sum_{p=1}^P \frac{(y_{p+1} - y_p)(x_p - x_{p-1}) - (y_p - y_{p-1})(x_{p+1} - x_p)}{[\xi(x_{p+1} - x_p) + \eta(y_{p+1} - y_p)][\xi(x_p - x_{p-1}) + \eta(y_p - y_{p-1})]} \exp\left[\pm i \frac{k}{f} (\xi x_p + \eta y_p)\right], \quad (7.66)$$

where $k = 2\pi/\lambda$ is the wave number.

Fraunhofer diffraction of a plane wave on the DOE with the form of a regular polygon and a piecewise constant microrelief

Consider the diffractive optical element having the shape of a regular polygon $\Omega = A_0 A_1 \dots A_{P-1}$, inscribed in a circle of radius R and containing the origin O . Then,

$$\Omega = \bigcup_{p=0}^{P-1} \Omega_p, \text{ where } \Omega_p \text{ are the triangles } OA_p A_{p+1}, \text{ and each vertex has the coordinates}$$

of A_p (Fig. 7.20)

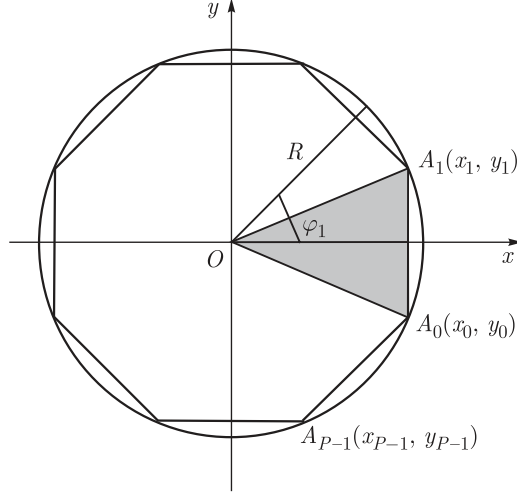


Fig. 7.20. DOE with the aperture in the form of a regular polygon.

$$\begin{cases} x_p = R \cos\left(\varphi_p - \frac{\pi}{P}\right), \\ y_p = R \sin\left(\varphi_p - \frac{\pi}{P}\right), \end{cases} \quad \varphi_p = \frac{2\pi}{P} p. \quad (7.67)$$

Let the depth of the microrelief inside of each triangle Ω_p be constant, then inside Ω_p and the complex transmission function of the DOE is constant:

$$\tau(x, y) = \exp(i\Psi_p). \quad (7.68)$$

Then, using the equation for the polygonal aperture, we can obtain an expression for the complex amplitude, which describes the Fraunhofer diffraction of a plane wave length λ at a DOE (Figure 7.20):

$$\begin{aligned} E(\xi, \eta) = & \frac{if}{2\pi k} R^2 \sin\left(\frac{2\pi}{P}\right) \sum_{p=0}^{P-1} \frac{\exp(i\Psi_p)}{(\xi x_{p+1} + \eta y_{p+1})(\xi x_p + \eta y_p)} - \\ & - \frac{if}{2\pi k} R^2 \sin\left(\frac{2\pi}{P}\right) \sum_{p=0}^{P-1} \frac{\exp(i\Psi_p) \exp\left[-i \frac{k}{f} (\xi x_p + \eta y_p)\right]}{\left[\xi(x_{p+1} - x_p) + \eta(y_{p+1} - y_p)\right](\xi x_p + \eta y_p)} + \\ & + \frac{if}{2\pi k} R^2 \sin\left(\frac{2\pi}{P}\right) \sum_{p=0}^{P-1} \frac{\exp(i\Psi_p) \exp\left[-i \frac{k}{f} (\xi x_{p+1} + \eta y_{p+1})\right]}{(\xi x_{p+1} + \eta y_{p+1})\left[\xi(x_{p+1} - x_p) + \eta(y_{p+1} - y_p)\right]}. \end{aligned} \quad (7.69)$$

In the transition to the polar coordinates instead of (7.68) we obtain the following expression:

$$E(\rho, \theta) = \frac{if}{2\pi k \rho^2} \sin\left(\frac{2\pi}{P}\right) \sum_{p=0}^{P-1} \frac{\exp(i\Psi_p)}{\cos\left(\varphi_p + \frac{\pi}{P} - \theta\right) \cos\left(\varphi_p - \frac{\pi}{P} - \theta\right)} +$$

$$+ \frac{if}{2\pi k \rho^2} \cos \frac{\pi}{P} \sum_{p=0}^{P-1} \left[\frac{\exp(i\Psi_p)}{\sin(\varphi_p - \theta)} - \frac{\exp(i\Psi_{p-1})}{\sin(\varphi_{p-1} - \theta)} \right] \frac{\exp\left[-i \frac{kR\rho}{f} \cos\left(\varphi_p - \frac{\pi}{P} - \theta\right)\right]}{\cos\left(\varphi_p - \frac{\pi}{P} - \theta\right)}.$$
(7.70)

In the case of quantized SPP, i.e. $\Psi_p = n\varphi_p$, from (7.69) we get:

$$E_n^P(\rho, \theta) = \frac{if}{2\pi k \rho^2} \sin\left(\frac{2\pi}{P}\right) \sum_{p=0}^{P-1} \frac{\exp(in\varphi_p)}{\cos\left(\varphi_p + \frac{\pi}{P} - \theta\right) \cos\left(\varphi_p - \frac{\pi}{P} - \theta\right)} +$$

$$+ \frac{if}{2\pi k \rho^2} \cos \frac{\pi}{P} \sum_{p=0}^{P-1} \left[\frac{\exp(in\varphi_p)}{\sin(\varphi_p - \theta)} - \frac{\exp(in\varphi_{p-1})}{\sin(\varphi_{p-1} - \theta)} \right] \frac{\exp\left[-i \frac{kR\rho}{f} \cos\left(\varphi_p - \frac{\pi}{P} - \theta\right)\right]}{\cos\left(\varphi_p - \frac{\pi}{P} - \theta\right)}.$$
(7.71)

Figure 7.21 shows a picture of the Fraunhofer diffraction of a plane wave on a continuous SPP limited by a circular aperture, obtained by the mean sum of Bessel functions [66]:

$$E_n(\rho, \theta) = \frac{(-i)^{n+1} k \exp(in\theta)}{f \bar{\rho}^2} \begin{cases} n \left[1 - J_0(y) - 2 \sum_{m=1}^{(n-2)/2} J_{2m}(y) \right] - y J_{n-1}(y), n = 2m, \\ n \left[\int_0^y J_0(t) dt - 2 \sum_{m=1}^{(n-1)/2} J_{2m-1}(y) \right] - y J_{n-1}(y), n = 2m+1, \end{cases}$$
(7.72)

where $y = R\bar{\rho} = kR\rho/f$, $J_n(x)$ is the Bessel function of the n -th order

$$\int_0^y J_0(t) dt = \frac{y}{2} \left\{ \pi J_1(y) H_0(y) + J_0(y) [2 - \pi H_1(y)] \right\},$$
(7.73)

$H_{0,1}(y)$ is the Struve function of zero and first orders.

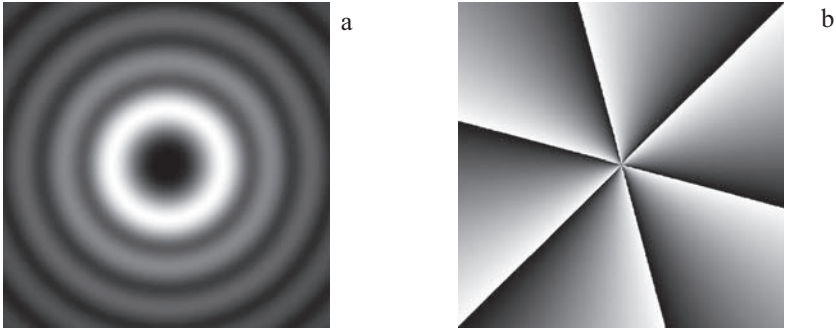


Fig. 7.21. Fraunhofer diffraction pattern of a plane wave on a continuous restricted SPP: amplitude (a) and phase (b).

Table 7.1

The number of sectors	SD
18	19.1411
30	1.9003
42	0.1320
54	0.0479

Table 7.2

Number of SPP	Minimum number of sectors
2	19
4	25
6	29
8	35
10	39

In the calculation we used the following parameters: wavelength 633 nm, the focal length of the spherical lens 150 mm, the radius of the aperture 2 mm, the order of the SPP 6.

Figure 7.22 shows the Fraunhofer diffraction pattern of a plane wave on a quantized limited TFP, obtained by the formula (7.71).

Table 7.1 shows the dependence of the standard deviation of the Fraunhofer diffraction pattern of a plane wave on a limited quantized spiral phase plate on the diffraction patterns from a limited continuous SPP for different numbers of sectors.

Table 7.2 shows, for several numbers of SPP, the minimum number of sectors of the multilevel SPP, in which the standard deviation of the Fraunhofer diffraction pattern from the diffraction pattern for continuous SPP does not exceed 2%.

7.4. Helical conical axion

The spiral phase plate is the simplest optical element intended to generate wave fronts with a helical phase singularity. The transmission function of the SPP has only one parameter – topological charge n . By varying it we can change the radius

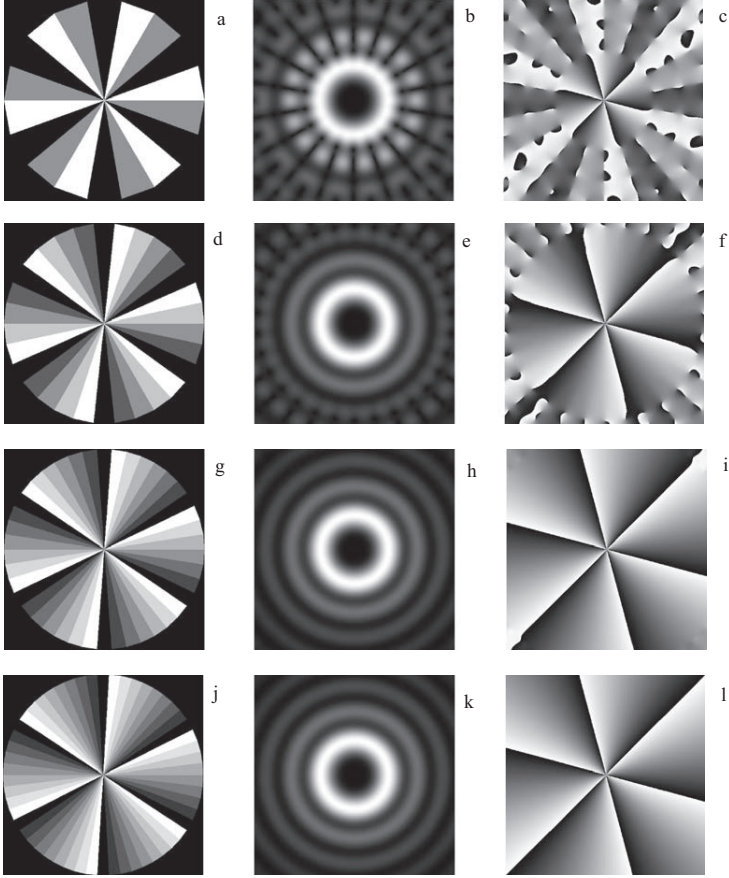


Fig. 7.22. Fraunhofer diffraction patterns of a plane wave on a quantized limited SPP: DOE phase (a, d, g, j), amplitude (b, d, h, k) and phase (c, f, i, l) in the zone of Fraunhofer diffraction. The number of sectors: 18 (a, b, c), 30 (d, e, f), 42 (g, h, i), 54 (j, k, l).

of the main ring of the diffraction pattern, however, to control other properties of the beam we do not have enough degrees of freedom. This leads to the idea of using combined optical elements which also include SPP. The simplest such element is a helical axicon whose phase depends linearly on both the angular and radial polar coordinates.

7.4.1. Diffraction of Gaussian beam in a helical axicon limited

Consider the scalar paraxial diffraction of a collimated Gaussian beam with a complex amplitude

$$E_0(r) = \exp\left(-\frac{r^2}{w^2}\right). \quad (7.74)$$

The helical axicon (HA), which in the approximation of a thin transparent is described by the transmission function of the form

$$\tau_n(r, \varphi) = \exp(i\alpha r + in\varphi), \quad (7.75)$$

where w is the Gaussian beam waist radius, (r, φ) are the polar coordinates in the plane of the HA at $z = 0$, z is the optical axis, α is the axicon parameter; $n = 0, \pm 1, \pm 2, \dots$, is the number of SPPP.

Then paraxial diffraction of the wave (7.74) on HA (7.75) is described by the Fresnel transformation:

$$F_n(\rho, \theta, z) = -\frac{ik}{2\pi z} \exp\left(ikz + \frac{ik\rho^2}{2z}\right) \times \int_0^R \int_0^{2\pi} \exp\left[-\frac{r^2}{w^2} + i\alpha r + in\varphi + \frac{ikr^2}{2z} - \frac{ik}{z} \rho r \cos(\varphi - \theta)\right] r dr d\varphi, \quad (7.76)$$

where (ρ, θ) are the polar coordinates in the plane z (z is the optical axis), $k = 2\pi/\lambda$ is the wavenumber. Using the background integral [53]

$$\int_0^\infty x^{\lambda+1} \exp(-px^2) J_\nu(cx) dx = \frac{c^\nu p^{-(\nu+\lambda+2)/2}}{2^{\nu+1} \nu!} \Gamma\left(\frac{\nu+\lambda+2}{2}\right) {}_1F_1\left[\frac{\nu+\lambda+2}{2}, \nu+1, -\left(\frac{c}{2\sqrt{p}}\right)^2\right], \quad (7.77)$$

instead of (7.76) we get:

$$F_n(\rho, \theta, z) = \frac{(-i)^{n+1} k}{z} \exp\left[in\theta + ikz + \frac{ik\rho^2}{2z}\right] \left(\frac{k\rho}{2z}\right)^n \frac{\gamma^{-(n+2)/2}}{2^{n+1} n!} \times \sum_{m=0}^\infty \frac{(i\alpha)^m \gamma^{-m/2}}{m!} \Gamma\left(\frac{m+n+2}{2}\right) {}_1F_1\left[\frac{m+n+2}{2}, n+1, -\left(\frac{k\rho}{2z\sqrt{\gamma}}\right)^2\right], \quad (7.78)$$

where $\gamma = 1/w^2 - ik/(2z)$, ${}_1F_1(a, b, x)$ is the degenerate or confluent hypergeometric function:

$${}_1F_1(a, b, x) = \sum_{m=0}^\infty \frac{(a)_m x^m}{(b)_m m!}, \quad (7.79)$$

$(a)_m = \Gamma(a+m)/\Gamma(a)$, $(a)_0 = 1$, and $\Gamma(x)$ is the gamma function.

From the expression (7.78) it follows that the diffraction pattern is a set of concentric rings. When $\rho = 0$ the intensity in the centre of the diffraction pattern at any $n \neq 0$ zero. Since the complex amplitude (7.78) depends on the combination of variables $k\rho/(2z\sqrt{\gamma})$ then the radii ρ_l of the local maxima and minima of the diffraction pattern must satisfy the following expression:

$$\rho_l = \frac{wz\alpha_l}{z_0} \left(1 + \frac{z_0^2}{z^2} \right)^{1/4}, \quad (7.80)$$

where α_l is a constant depending only on the number of ring $l = 1, 2, \dots$ of diffraction patterns and the parameter α , $z_0 = kw^2/2$ is the Rayleigh length.

At $\alpha = 0$ (i.e. no axicon), from (7.78) we obtain the relationship for the complex amplitude of Fresnel diffraction of the Gaussian beam on the SPP:

$$\begin{aligned} F_n(\rho, \theta, z, \alpha = 0) &= \frac{(-i)^{n+1} k}{z} \exp \left[i(n\theta + kz) + \frac{ik\rho^2}{2z} \right] \left(\frac{k\rho}{2z} \right)^n \times \\ &\times \frac{\gamma^{-(n+2)/2}}{2^{n+1} n!} \Gamma \left(\frac{n+2}{2} \right) {}_1F_1 \left[\frac{n+2}{2}, n+1, -\left(\frac{k\rho}{2z\sqrt{\gamma}} \right)^2 \right]. \end{aligned} \quad (7.81)$$

Given the connection between the hypergeometric and Bessel functions

$$J_{(n-1)/2}(x) = \frac{\left(\frac{x}{2} \right)^{(n-1)/2} \exp(-ix)}{\Gamma \left(\frac{n-1}{2} \right)} {}_1F_1 \left(\frac{n}{2}, n; 2ix \right) \quad (7.82)$$

and the recurrence relation for the hypergeometric functions

$${}_1F_1 \left(\frac{n}{2}, n+1; 2ix \right) = \left(i \frac{d}{dx} + 2 \right) {}_1F_1 \left(\frac{n}{2}, n; 2ix \right), \quad (7.83)$$

we can replace (7.81) to obtain a well-known relation for the Fresnel diffraction of Gaussian beam on the SPP [28, 52]:

$$\begin{aligned} E_n(\rho, \theta, z, \alpha = 0) &= \frac{(-i)^{n+1} \sqrt{\pi}}{2} \left(\frac{z_0}{z} \right)^2 \left(\frac{\rho}{w} \right) \left[1 + \left(\frac{z_0}{z} \right)^2 \right]^{-3/4} \times \\ &\times \exp \left[i \frac{3}{2} \tan^{-1} \left(\frac{z_0}{z} \right) - i \frac{k\rho^2}{2R_0(z)} + i \frac{k\rho^2}{2z} - \frac{\rho^2}{w^2(z)} + in\theta + ikz \right] \times \\ &\times \left\{ I_{\frac{n-1}{2}} \left[\rho^2 \left(\frac{1}{w^2(z)} + \frac{ik}{2R_0(z)} \right) \right] - I_{\frac{n+1}{2}} \left[\rho^2 \left(\frac{1}{w^2(z)} + \frac{ik}{2R_0(z)} \right) \right] \right\}, \end{aligned} \quad (7.84)$$

where $w^2(z) = 2w^2[1 + (z/z_0)^2]$, $R_0(z) = 2z[1 + (z/z_0)^2]$, $I_\nu(x)$ is the Bessel function of second kind and ν -th order.

When $z \rightarrow \infty$ ($z \gg z_0$) the expression (7.78) yields the following formula for the complex amplitude of Fraunhofer diffraction of the Gaussian beam on the HA ($\gamma = 1/w^2$)

$$F_n(\rho, \theta, z \rightarrow \infty) = \frac{(-i)^{n+1} z_0}{2^n n! z} \exp\left(in\theta + ikz + \frac{ik\rho^2}{2z}\right) \left(\frac{z_0 \rho}{zw}\right)^n \times \sum_{m=0}^{\infty} \frac{(i\alpha w)^m}{m!} \Gamma\left(\frac{m+n+2}{2}\right) {}_1F_1\left[\frac{m+n+2}{2}, n+1, -\left(\frac{z_0 \rho}{zw}\right)^2\right]. \quad (7.85)$$

At $\alpha = 0$ (i.e. no axicon) and $z \rightarrow \infty$ ($z \gg z_0$) from (7.78) follows the expression for the complex amplitude of Fraunhofer diffraction of the Gaussian beam on the SPP:

$$F_n(\rho, \theta, z \rightarrow \infty, \alpha = 0) = \frac{(-i)^{n+1} z_0}{2^n n! z} \exp\left(in\theta + ikz + \frac{ik\rho^2}{2z}\right) \left(\frac{z_0 \rho}{zw}\right)^n \Gamma\left(\frac{n+2}{2}\right) {}_1F_1\left[\frac{n+2}{2}, n+1, -\left(\frac{z_0 \rho}{zw}\right)^2\right]. \quad (7.86)$$

It is interesting to compare the expression (7.86) with the complex amplitude of Fraunhofer diffraction of a restricted plane wave of radius R on the SPP, when the focal length of the spherical lens is equal to f [67]:

$$E_n(\rho, \theta) = \frac{(-i)^{n+1} \exp(in\theta + ikz)}{(n+2)n!} \left(\frac{kR^2}{f}\right) \left(\frac{kR\rho}{2f}\right)^n {}_1F_2\left[\frac{n+2}{2}, \frac{n+4}{2}, n+1, -\left(\frac{kR\rho}{2f}\right)^2\right], \quad (7.87)$$

where ${}_1F_2(a, b, c, x)$ is the hypergeometric function:

$${}_1F_2(a, b, c, x) = \sum_{m=0}^{\infty} \frac{(a)_m x^m}{(b)_m (c)_m m!}. \quad (7.88)$$

Figure 7.23 shows the calculated distribution of the amplitude $|F_n(\rho, \theta)|$ in relative units as a function of the radial variable. These curves represent the radial profile of the Fresnel diffraction pattern ($z = 200$ mm) of the Gaussian beam with the waist radius $w = 1$ mm and a wavelength $\lambda = 633$ nm on the HA ($n = 8$) with parameter $\alpha = 0$ mm⁻¹ (a), $\alpha = 20$ mm⁻¹ (b), $\alpha = 50$ mm⁻¹ (c).

Figure 7.23 shows that the radius of the main peak of the amplitude increases with increasing values of α .

Figure 7.24 shows two calculated radial Fresnel diffraction patterns (amplitude $|F_n(\rho, \theta)|$) for a Gaussian beam ($w = 1$ mm, $\lambda = 633$ nm) for HA ($n = 8$) with parameter $\alpha = 20$ mm⁻¹ at a distance $z = 400$ mm (a) and $z = 500$ mm (b). From Fig. 7.24 it can be seen that with increasing distance z the radius of the first bright ring in the diffraction pattern, characterized by the maximum amplitude, also increases. Comparing Figures 7.23 and 7.24 gives reason to conclude that the radius of the first ring can be changed either by changing the parameter α of the axicon at

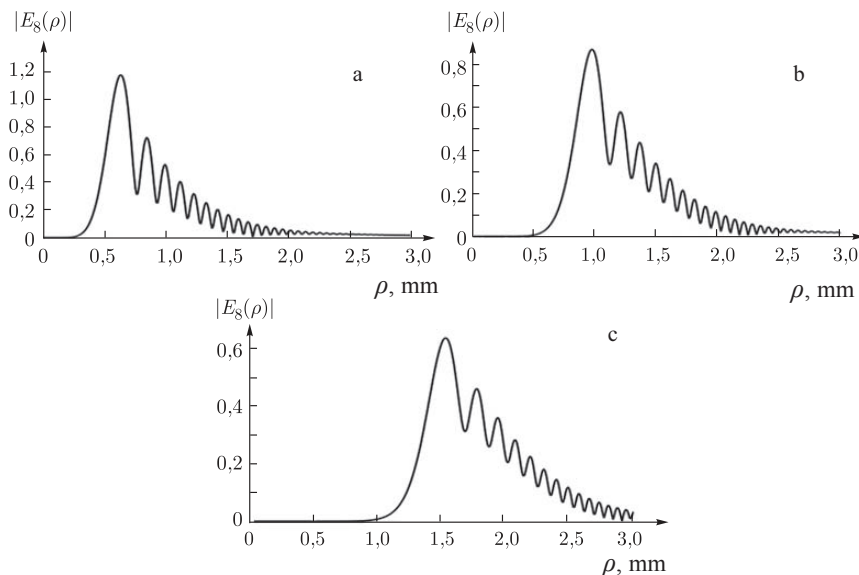


Fig. 7.23. The radial profile of the Fresnel diffraction pattern (the amplitude at a distance $z = 200$ mm) for a Gaussian beam ($\lambda = 633$ nm, $w = 1$ mm) on the HA ($n = 8$): $\alpha = 0 \text{ mm}^{-1}$ (a), $\alpha = 20 \text{ mm}^{-1}$ (b), $\alpha = 50 \text{ mm}^{-1}$ (c).

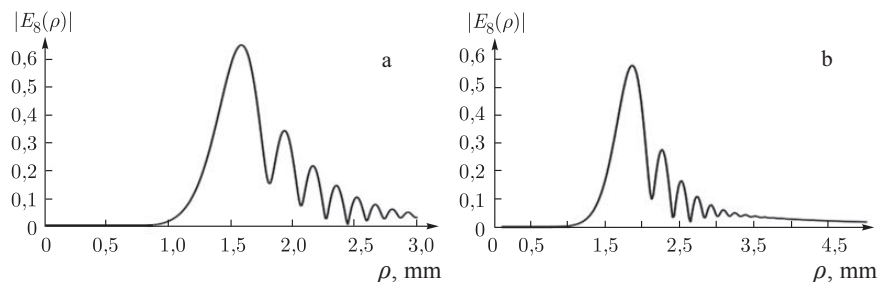


Fig. 7.24. The radial profile of the Fresnel diffraction pattern (amplitude $|F_n(\rho, \theta)|$) of a Gaussian beam ($\lambda = 633$ nm, $w = 1$ mm) on the HA ($n = 8$, $\alpha = 20 \text{ mm}^{-1}$): $z = 400$ mm (a), $z = 500$ mm (b).

a constant distance z , or by changing the distance z from the axicon to the plane of observation. The difference will be in the amount of the peripheral rings (sidelobes) in the diffraction pattern. From Fig. 7.23 it can be seen that 13 peripheral diffraction rings are stacked in a radial range from 1.5 mm to 3 mm. At the same time in Fig. 7.24a in the same radial range from 1.5 mm to 3 mm there are only seven lateral lobes, despite the fact that the radius of the first ring is the same in both patterns.

Figure 7.25 shows two calculated radial Fresnel diffraction pattern (the amplitude $|F_n(\rho, \theta)|$ at a distance $z = 200$ mm) Gaussian beam ($w = 1$ mm, $\lambda = 633$ nm) at HA ($\alpha = 20 \text{ mm}^{-1}$) of different orders n : 40 (a) and 50 (b).

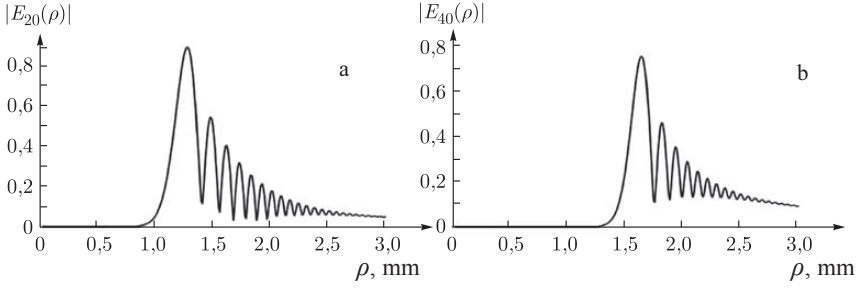


Fig. 7.25. The radial profile of the Fresnel diffraction pattern (amplitude $|F_n(\rho, \theta)|$ at distance $z = 200$ mm) of a Gaussian beam ($\lambda = 633$ nm, $w = 1$ mm) on the HA ($\alpha = 20$ mm $^{-1}$): a) $n = 20$ mm, b) $n = 40$.

Figure 7.25 shows that the radius of the first ring in the diffraction pattern can be changed by varying both the order of HA n , and the parameter α . Note, however, that in the case of Fig. 7.25, in addition to an increase in the radius of the first ring, an increase of order n leads to thinning of the first ring, to a larger number of peripheral rings and an increased contrast of the rings.

7.4.2. Diffraction of a restricted plane wave on a helical axicon

Diffraction of an unbounded plane wave is considered in [68]. Fraunhofer diffraction of a plane wave by a helical axicon of a finite radius with the transmission function $\text{circ}(r/R)\exp(iar)$ is described by the following expression:

$$F(\bar{\rho}) = \int_0^R \exp(iar) J_n\left(\frac{k}{f} \bar{\rho} r\right) r dr. \quad (7.89)$$

We consider the integral:

$$I = \int_0^R \exp(iar) J_n(\rho r) r dr, \quad \rho = \frac{k}{f} \bar{\rho}. \quad (7.90)$$

Using the integral representation of Bessel functions

$$J_n(x) = \frac{(-i)^n}{2\pi} \int_0^{2\pi} \exp(in\varphi) \exp(ix \cos \varphi) d\varphi, \quad (7.91)$$

we obtain:

$$I = \frac{(-i)^{n+2}}{2\pi} \frac{\partial}{\partial \alpha} \left[\exp(i\alpha R) \int_0^{2\pi} \exp(in\varphi) \frac{\exp(iR\rho \cos \varphi)}{\alpha + \rho \cos \varphi} d\varphi - \int_0^{2\pi} \frac{\exp(in\varphi) d\varphi}{\alpha + \rho \cos \varphi} \right]. \quad (7.92)$$

Using the known relation for the Bessel functions

$$\exp(ix \cos \varphi) = \sum_{m=-\infty}^{+\infty} i^m \exp(-im\varphi) J_m(x), \quad (7.93)$$

instead of the integral (7.92) we can obtain an expression for the diffraction in the form of a series:

$$I = \frac{(-i)^n}{2\pi} \frac{\partial I_1^n}{\partial \alpha} - \frac{\exp(i\alpha R)}{2\pi} \sum_{m=-\infty}^{+\infty} i^m \left(iR I_1^m + \frac{\partial I_1^m}{\partial \alpha} \right) J_{m+n}(R\rho), \quad (7.94)$$

where

$$I_1^n = \int_0^{2\pi} \frac{\exp(in\varphi) d\varphi}{\alpha + \rho \cos \varphi}. \quad (7.95)$$

The integrals (7.95) and their derivatives are computed by applying the theory of residues. The expressions for the integrals I_1^n and $\partial I_1^n / \partial \alpha$ are given below.

Case 1. $0 < \rho < |\alpha|$.

$$I_1^n = \frac{2\pi \operatorname{sgn} \alpha}{\sqrt{\alpha^2 - \rho^2}} \left(\frac{-\alpha + \operatorname{sgn} \alpha \sqrt{\alpha^2 - \rho^2}}{\rho} \right)^{|n|}, \quad (7.96)$$

$$\frac{\partial I_1^n}{\partial \alpha} = -2\pi \operatorname{sgn} \alpha \left(\frac{-\alpha + \operatorname{sgn} \alpha \sqrt{\alpha^2 - \rho^2}}{\rho} \right)^{|n|} \frac{\alpha + \operatorname{sgn} \alpha |n| \sqrt{\alpha^2 - \rho^2}}{(\alpha^2 - \rho^2)^{\frac{3}{2}}}. \quad (7.97)$$

Case 2. $\rho > |\alpha|$:

$$I_1^n = \frac{\pi i}{\sqrt{\rho^2 - \alpha^2}} \left[\left(\frac{-\alpha - i\sqrt{\rho^2 - \alpha^2}}{\rho} \right)^{|n|} - \left(\frac{-\alpha + i\sqrt{\rho^2 - \alpha^2}}{\rho} \right)^{|n|} \right] = \pi i \frac{\chi^{*|n|} - \chi^{|n|}}{\sqrt{\rho^2 - \alpha^2}}, \quad (7.98)$$

where $\chi = \left[-\alpha + i(\rho^2 - \alpha^2)^{1/2} \right] / \rho$.

$$\frac{\partial I_1^n}{\partial \alpha} = \pi i \left[\alpha (\rho^2 - \alpha^2)^{-\frac{3}{2}} (\chi^{*|n|} - \chi^{|n|}) - i(\rho^2 - \alpha^2)^{-1} |n| (\chi^{*|n|} + \chi^{|n|}) \right]. \quad (7.99)$$

Diffraction of a restricted plane wave on a spiral phase plate

We obtain a formula for the Fraunhofer diffraction of a plane wave on a limited spiral phase plate (i.e. $\alpha = 0$). We shall also assume that $n \geq 0$, as for $n < 0$ it is

sufficient to multiply the complex amplitude in the output plane by $(-1)^n$ (it can be seen from (7.89)).

At $\alpha = 0$ the expressions for the integrals I_1^n and $\partial I_1^n / \partial \alpha$ can be significantly simplified:

$$I_1^n = \begin{cases} 0, n = 2m, \\ -2\pi i^{|2m+1|+1} \rho^{-1}, n = 2m+1; \end{cases} \quad (7.100)$$

$$\frac{\partial I_1^n}{\partial \alpha} = \begin{cases} 2\pi i^{|n|} |n| \rho^{-2}, n = 2m, \\ 0, n = 2m+1. \end{cases} \quad (7.101)$$

Substituting these expressions in (7.94), dividing the sum $m \in (-\infty, +\infty)$ by the sums $m \in [0, +\infty)$ and $m \in (-\infty, -1]$, to get rid of the modules, using the recurrence relation for Bessel functions $2\nu J_\nu(z) = z[J_{\nu-1}(z) + J_{\nu+1}(z)]$ and, given that $\lim_{\nu \rightarrow \infty} J_\nu(z) = 0$ we can reduce (7.94) to the following form:

$$I = \frac{(-i)^n}{2\pi} \frac{\partial I_1^n}{\partial \alpha} + n\rho^{-2} \left[\sum_{m=0}^{+\infty} J_{2m+n}(R\rho) - \sum_{m=1}^{+\infty} J_{n-2m}(R\rho) \right] - R\rho^{-1} J_{n-1}(R\rho). \quad (7.102)$$

For even $n \geq 0$:

$$I = n\rho^{-2} - n\rho^{-2} \left[J_0(R\rho) + 2 \sum_{m=1}^{n/2-1} J_{2m}(R\rho) \right] - R\rho^{-1} J_{n-1}(R\rho). \quad (7.103)$$

Multiplying by k/f and substituting ρ for $k\rho/f$, we obtain a formula for the Fraunhofer diffraction of a plane wave on a restricted SPP of the even non-negative integer order n :

$$F(\rho) = \frac{f}{k\rho^2} \left\{ n \left[1 - J_0\left(\frac{k}{f} R\rho\right) \right] - \frac{k}{f} R\rho J_{n-1}\left(\frac{k}{f} R\rho\right) - 2n \sum_{m=1}^{n/2-1} J_{2m}\left(\frac{k}{f} R\rho\right) \right\}. \quad (7.104)$$

For odd $n, n > 0$:

$$I = n\rho^{-2} \left[\int_0^{R\rho} J_0(x) dx - 2 \sum_{m=1}^{(n-1)/2} J_{2m-1}(R\rho) \right] - R\rho^{-1} J_{n-1}(R\rho). \quad (7.105)$$

Multiplying by k/f and substituting ρ for $k\rho/f$, we obtain a formula for the Fraunhofer diffraction of a plane wave on a restricted SPP of the positive odd integer order n :

$$F(\rho) = \frac{f}{k\rho^2} \left[n \int_0^{\frac{k}{f}R\rho} J_0(x) dx - 2n \sum_{m=1}^{(n-1)/2} J_{2m-1}\left(\frac{k}{f}R\rho\right) - \frac{k}{f}R\rho J_{n-1}\left(\frac{k}{f}R\rho\right) \right]. \quad (7.106)$$

The use of a conical axicon provides an additional degree of freedom (the parameter α) as compared to a plane wave. For example, we can achieve a smooth radial distribution of the amplitude.

Figures 7.26 and 7.27 show the results of numerical simulation of Fraunhofer diffraction of a plane wave by a helical axicon with a finite radius. We used the following settings:

- Wavelength: $\lambda = 633$ nm.
- Focal length of spherical lens: $f = 140$ mm.
- The order of the SPP: $n = 4$.
- Parameter axicon: $\alpha = 0$ mm⁻¹ (i.e. no axicon) (a) and $\alpha = 1$ mm⁻¹ (b).
- Aperture radius: $R = 2$ mm.

It is seen that the graph in Fig. 7.26b, obtained using an axicon, is ‘smoother’.

If we increase the value of the axicon parameter α , then increase of the value of the radial coordinate ρ is accompanied by an increase in the number of ‘lobes’ (Fig. 7.28).

Experiments on the formation of a ring of light with the help of the CA are given in [58].

Thus, the use of the helical axicon raises the possibility of formation of optical vortices with desired characteristics. This is of practical importance for the problems of nanophotonics, in particular the optical manipulation of micro- and nano-objects. Due to the pressure of light these objects tend to be drawn into the area with the greatest intensity, but the presence of side lobes in the diffraction pattern shown in Figs. 7.26a and 7.27a means that an object can be drawn into the side instead of the main ring. The values of the radius and the speed of rotation of

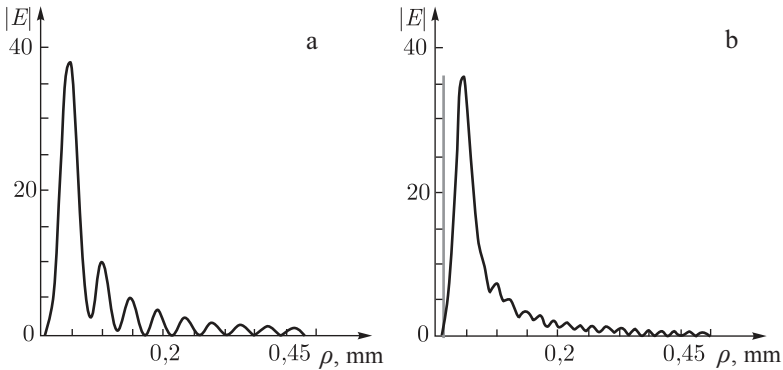


Fig. 7.26. The result of numerical simulation of Fraunhofer diffraction of a plane wave on a helical axicon with a finite radius (dependence of the modulus of the amplitude on the radial coordinate) without the axicon $\alpha = 0$ (a) and with an axicon $\alpha = 1$ mm⁻¹ (b).

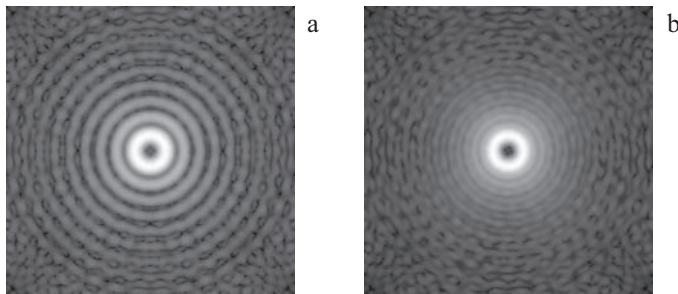


Fig. 7.27. The result of numerical simulation of Fraunhofer diffraction of a plane wave on a helical axicon finite radius (two-dimensional diffraction pattern) with no axicon $\alpha = 0$ (a) and with an axicon $\alpha = 1 \text{ mm}^{-1}$ (b).

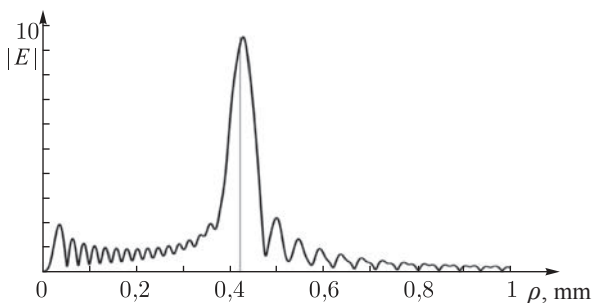


Fig. 7.28. The result of numerical simulation of Fraunhofer diffraction of a plane wave on a helical axicon with a finite radius at the axicon parameter $\alpha = 30 \text{ mm}^{-1}$.

the object will differ from the target. The use of an axicon for the formation of the diffraction patterns, shown in Figs. 7.26b and 7.27b, is designed to eliminate this problem.

7.5. Helical logarithmic axicon

7.5.1. General theory of hypergeometric laser beams

Let us consider a light field with the initial function of the complex transmittance of the form:

$$E_{\gamma nm}(r, \varphi) = \frac{1}{2\pi} \left(\frac{r}{w} \right)^m \exp \left(-\frac{r^2}{2\sigma^2} + i\gamma \ln \frac{r}{w} + in\varphi \right), \quad (7.107)$$

where (r, φ) - polar coordinates in the initial plane ($z = 0$), w and γ are the actual parameters of the logarithmic axicon, σ is the Gaussian beam waist radius, n is the integer order of a spiral phase plate, m is a parameter. The complex amplitude (7.107) describes a light field with infinite energy and a singularity at $r = 0$ and $m < 0$. Despite this, in any transverse plane at a distance z from the initial plane the complex amplitude of the light field, generated by the function (7.107), will not have any singularities and will be final.

In the paraxial propagation of the light field (7.107), its complex amplitude at a distance z will be determined by the Fresnel transform, which in polar coordinates has the form:

$$E(\rho, \theta, z) = -\frac{ik}{2\pi z} \iint_{R^2} E(r, \varphi, 0) \exp\left\{\frac{ik}{2z}[\rho^2 + r^2 - 2\rho r \cos(\varphi - \theta)]\right\} r dr d\varphi. \quad (7.108)$$

We have the reference integral:

$$\int_0^\infty x^{\alpha-1} \exp(-px^2) J_\nu(cx) dx = c^\nu p^{-\frac{\nu+\alpha}{2}} 2^{-\nu-1} \Gamma\left(\frac{\nu+\alpha}{2}\right) \Gamma^{-1}(\nu+1) {}_1F_1\left(\frac{\nu+\alpha}{2}, \nu+1, -\frac{c^2}{4p}\right), \quad (7.109)$$

where ${}_1F_1(a, b, x)$ is the confluent hypergeometric function or Kummer's function $\Gamma(x)$ is the gamma function.

Then the transformation from the Fresnel (7.108) has the form:

$$E_{\gamma nm}(\rho, \theta, z) = \frac{(-i)^{n+1}}{2\pi n!} \left(\frac{z_0}{zq^2}\right) \left(\frac{\sqrt{2}\sigma}{wq}\right)^{m+i\gamma} \left(\frac{k\sigma\rho}{\sqrt{2}qz}\right)^n \exp\left(\frac{ik\rho^2}{2z} + in\theta\right) \times \\ \times \Gamma\left(\frac{n+m+2+i\gamma}{2}\right) {}_1F_1\left(\frac{n+m+2+i\gamma}{2}, n+1, -\left(\frac{k\sigma\rho}{\sqrt{2}qz}\right)^2\right), \quad (7.110)$$

where $z_0 = k\sigma^2$, $q = (1-iz_0/z)^{1/2}$. Laser beams with a complex amplitude (7.110) are termed hypergeometric beams (HG-beams).

The modulus of the complex amplitude (7.110) is proportional to Kummer's function:

$$|E_{\gamma nm}(\rho, \theta, z)| \sim x^{\frac{n}{2}} {}_1F_1(a, b, -x), \quad (7.111)$$

where x is a complex argument:

$$x = \left(\frac{k\sigma\rho}{\sqrt{2}qz}\right)^2. \quad (7.112)$$

Since the Kummer function is represented in the form of a series:

$${}_1F_1(a, b, -x) = \sum_{l=0}^{\infty} C_l (-1)^l x^l, \quad (7.113)$$

where

$$C_l = \frac{\Gamma(a+l)}{\Gamma(a)} \frac{\Gamma(b)}{\Gamma(b+l)l!}, \quad (7.114)$$

then

$$\begin{aligned}
 x^l &= \left(\frac{k\sigma\rho}{\sqrt{2}z} \right)^l \frac{1}{q^{2l}} = \left\{ \frac{1}{q^{2l}} = \frac{1}{(1-iz_0/z)^l} = \left[\frac{\exp\left(i \arctg \frac{z_0}{z}\right)}{\sqrt{1+z_0^2/z^2}} \right]^l \right\} = \\
 &= \left[\frac{k\sigma\rho}{\sqrt{2}z(1+z_0^2/z^2)^{1/4}} \right]^{2l} \exp\left(i \arctg \frac{z_0}{z}\right). \tag{7.115}
 \end{aligned}$$

Then

$${}_1F_1(a, b, -x) = \sum_{l=0}^{\infty} (-i)^l C_l \left[\frac{k\sigma\rho}{\sqrt{2}z(1+z_0^2/z^2)^{1/4}} \right]^{2l} \exp\left(-i \arctg \frac{z}{z_0}\right), \tag{7.116}$$

From (7.116) it follows that the function $|{}_1F_1|$ and the amplitude and phase of each term of the series varies with changes in z . This means that each ‘partial’ light field at $l = \text{const}$ in (7.116) will be distributed in space with its phase velocity determined by the factor $\exp[-il \arctg(z/z_0)]$. As a result of the longitudinal interference of all terms in (7.116), the modulus of function (7.116), and hence the modulus of the complex amplitude of the light field (7.110), will change its appearance during propagation.

Hypergeometric beams in the near zone

At $z \ll z_0 = k\sigma^2$ the dependence on σ is lost, as

$$\frac{1}{2\sigma^2} - \frac{ik}{2z} \approx -\frac{ik}{2z}. \tag{7.117}$$

Then $q \approx \sqrt{-iz_0/z}$, $q^2 \approx iz_0/z$, and

$$\begin{aligned}
 E_{\gamma nm}(\rho, \theta, z \ll z_0) &= \frac{(-i)^{\frac{n-m-i\gamma}{2}}}{2\pi n!} \left(\frac{k\rho^2}{2z} \right)^{\frac{m+i\gamma}{2}} \left(\frac{k\rho^2}{2z} \right)^{\frac{n}{2}} \exp\left(\frac{ik\rho^2}{2z} + i n \theta\right) \times \\
 &\times \Gamma\left(\frac{n+m+2+i\gamma}{2}\right) {}_1F_1\left(\frac{n+m+2+i\gamma}{2}, n+1, -\frac{ik\rho^2}{2z}\right). \tag{7.118}
 \end{aligned}$$

From (7.118) it follows that at $z \ll z_0$ the modulus of the complex amplitude $|E_{\gamma nm}(\rho, \theta, z \ll z_0)|$ will maintain its form and vary only on a large scale. Note that at $\sigma \rightarrow \infty$ (Gaussian beam is replaced by a flat unlimited bundle) and instead of (7.110) we obtain the equation (7.118) describing the paraxial mode beams, the generalized hypergeometric modes [66].

Hypergeometric beams in the far zone

At $z \gg z_0 = k\sigma^2 q = (1 - iz_0/z)^{1/2} \approx 1$. Then

$$E_{\gamma nm}(\rho, \theta, z \gg z_0) = \frac{(-i)^{n+1}}{2\pi n!} \left(\frac{z_0}{z}\right) \left(\frac{\sqrt{2}\sigma}{w}\right)^{m+i\gamma} \left(\frac{k\sigma\rho}{\sqrt{2}z}\right)^n \exp\left(\frac{ik\rho^2}{2z} + in\theta\right) \times \\ \times \Gamma\left(\frac{n+m+2+i\gamma}{2}\right) {}_1F_1\left(\frac{n+m+2+i\gamma}{2}, n+1, -\left(\frac{k\sigma\rho}{\sqrt{2}z}\right)^2\right). \quad (7.119)$$

The dependence of the diffraction pattern on z changes qualitatively. And in the near- and far-field zones the diffraction pattern has a set of concentric rings of light with increasing spatial frequency, since the distribution of the amplitude is proportional to ρ^2 . But in the near-field the diffraction pattern does not change (up to a factor) at a constant ratio ρ^2/z , while in the far field – at a constant ratio ρ/z . That is, in propagation of near-field the light ring radii grow more slowly than in the far-field: in the near-field the radii of the rings grow in proportion to $\sqrt{z}$, and in the far-field in proportion to z .

7.5.2. Hypergeometric modes

The Helmholtz equation, which describes the propagation of a non-paraxial monochromatic light wave in a homogeneous space permits eleven solutions with separable variables in different coordinate systems [69]. This means that there are light fields, which propagate without changing their structure. Examples are the well-known Bessel modes [70]. The paraxial analogue of the Helmholtz equation is the parabolic equation of Schrödinger type, which describes the propagation of paraxial optical fields. This equation permits seventeen solutions with separable variables in the coordinate systems [71]. Light fields, which are described by such solutions, retain their structure during propagation up to scale. Example include the well-known Hermite–Gaussian and Laguerre–Gaussian modes [71].

In recent years there has been a dramatic increase in the number of papers in which solutions with separable variables for the Helmholtz equation and Schrödinger were used in optics [72–80]. New non-paraxial light beams that retain their structure during the propagation were considered in [72–74]. These are parabolic bundles [72], Helmholtz–Gauss waves [73] and Laplace–Gauss waves [74]. New paraxial light beams that retain their structure up to scale were considered in [75–80]. These are Ince–Gaussian modes [75], elegant Ince–Gaussian beams [76], Hermite–Laguerre–Gaussian modes [77] and the pure optical vortices [28]. Some of these beams have been realized with laser resonators [77, 78], diffractive optical elements [28] and liquid crystal displays [79].

This section deals with another family of laser modes, which are an orthonormal basis and are solutions with separated variables of the paraxial parabolic equation in a cylindrical coordinate system. In this coordinate system, the Schrödinger equation except for solutions in the form of Bessel and Laguerre–Gaussian modes, also has

a solution in the form of confluent hypergeometric functions. These solutions are special cases of the considered hypergeometric beams of general form. Intensity distribution in the cross section of such beams is close to the intensity distribution for the Bessel modes. It is also a set of concentric light rings, but their intensity decreases with increasing radial variable as r^{-2} , i.e. faster than that for the Bessel modes. Like the Bessel modes, the hypergeometric modes have infinite energy. In contrast to the Bessel modes, the light ring radii of the hypergeometric modes increases with increasing longitudinal coordinate z as $\sqrt{z}$. Experiments with the generation of such laser modes using liquid crystal microdisplays are also described.

The complex amplitude of the paraxial optical field $E(r, \varphi, z)$ in a cylindrical coordinate system (r, φ, z) satisfies the equation of Schrödinger type:

$$\left(2ik \frac{\partial}{\partial z} + \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} \right) E(r, \varphi, z) = 0, \quad (7.120)$$

where $k = 2\pi/\lambda$ is the wavenumber of light with the wavelength λ . Equation (7.120) is satisfied by the functions that form an orthonormal basis:

$$E_{\gamma, n}(r, \varphi, z) = \frac{1}{2\pi n!} \left(\frac{z_0}{z} \right)^{\frac{1}{2}} \Gamma \left(\frac{n+1+i\gamma}{2} \right) \times \\ \times \exp \left[\frac{i\pi}{4} (3n+i\gamma-1) + \frac{i\gamma}{2} \ln \frac{z_0}{z} + in\varphi \right] x^{\frac{n}{2}} {}_1F_1 \left(\frac{n+1-i\gamma}{2}, n+1, ix \right), \quad (7.121)$$

where $-\infty < \gamma < \infty$, $n = 0, \pm 1, \pm 2, \dots$ are continuous and discrete parameters that affect the functions (7.121) and which will be called the mode numbers; $z_0 = kw^2/2$ is an analog of Rayleigh length, w is the mode parameter, similar to the radius of the Gaussian beam, although it has a different meaning here; $x = kr_z/(2z)$; $\Gamma(x)$ is the gamma function; ${}_1F_1(a, b, y)$ is the degenerate or confluent hypergeometric function [81]:

$${}_1F_1(a, b, y) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 t^{a-1} (1-t)^{b-a-1} \exp(yt) dt, \quad (7.122)$$

where $\text{Re}(b) > \text{Re}(a) > 0$. From (7.122) we see that ${}_1F_1(a, b, y)$ is an entire analytic function. In the case of (7.121) $\text{Re}(y) = 0$ and then (7.122) is a one-dimensional Fourier transform of a bounded function on the interval $[0, 1]$. According to Shannon's theorem asymptotically at $r \rightarrow \infty$ the modulation period of function (7.121) (i.e. the distance between adjacent maxima or minima) is 2π . For large values of the argument $x \gg 1$ we have the asymptotic behaviour $x^{n/2} |{}_1F_1[(n+1-i\gamma)/2, n+1, ix]| \approx 1/\sqrt{x}$. This behaviour of the modulus of the function (7.121) leads to a more rapid decline than that of the Bessel function. In addition, the zeros of the confluent hypergeometric functions are ${}_1F_1(a, b, y_{0m})$ are close to the zeros of Bessel functions $J_{b-1}(y_{b-1, m})$ [81]:

$$y_{0,m} \approx \left| \frac{y_{b-1,m}^2}{2b-4a} \right|. \quad (7.123)$$

The light beams (7.121), which will be called hypergeometric (HG) modes, can be generated using an optical element having a transmission function:

$$E_{\gamma,n}(\rho, \theta, z) = \frac{1}{2\pi} \left(\frac{w}{\rho} \right) \exp \left[i\gamma \ln \left(\frac{\rho}{w} \right) + in\theta \right]. \quad (7.124)$$

In illuminating the optical element (7.124), located in the plane $z = 0$, by an unbounded plane wave a light field with the complex amplitude (7.121) forms at distance z . The energy of the light fields (7.121) and (7.124) is unbounded, as in the the Bessel mode

$$E_{\beta,n}(r, \varphi, z) = J_n(\beta r) \exp \left[i \frac{\beta^2 z}{2k} + in\varphi \right], \quad (7.125)$$

which also satisfies (7.120). Therefore, to produce the mode (7.121) in practice, the optical element (7.124) should be limited by a circular aperture. At the same time, the mode (7.121) will form effectively at a finite distance $z_0 < R \operatorname{tg}(\gamma/R)$, where R is the large radius of the circular aperture.

In propagation, the light field (7.121) retains its structure and only its scale changes. The transverse intensity distribution of the HG mode (7.121) is a set of concentric light rings, whose radii satisfy the condition:

$$r_m = (\alpha_m z \lambda / \pi)^{1/2}, \quad (7.126)$$

where α_m is a constant depending on the number of rings m and the number of modes (γ, n) . Therefore, the ring radii increase with increasing z as $\sqrt{z}$. From the relation [81]:

$${}_1F_1 \left(\frac{n+1+i\gamma}{2}, n+1, -ix \right) = \exp(-ix) {}_1F_1 \left(\frac{n+1-i\gamma}{2}, n+1, ix \right) \quad (7.127)$$

it follows that the phase of the hypergeometric function is equal to $x/2$ (up to π):

$$\arg \left\{ {}_1F_1 \left(\frac{n+1+i\gamma}{2}, n+1, -ix \right) \right\} = -\frac{x}{2}. \quad (7.128)$$

Interestingly, this phase does not depend on the number of the mode (γ, n) . Then we can write the expression for the phase of the HG mode:

$$\arg \{ E_{\gamma,n}(r, \varphi, z) \} = \frac{\gamma}{2} \ln \frac{z}{z_0} + n\varphi + \frac{kr^2}{4z} + \frac{\pi}{4} (3n-1) + \arg \Gamma \left(\frac{n+1+i\gamma}{2} \right), \quad (7.129)$$

where the first term has the meaning of the ???Gow??? phase.